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Ein Schließungssatz für Inzidenz und Orthogonalität in Möbiusebenen

Von
GÜNTER EWALD in Mainz und Oberwolfach

Einleitung

Unter einer *Möbiusebene* verstehen wir mit W. BENZ [2] ein System von *Punkten* $A, B, \ldots$ und *Kreisen* $a, b, \ldots$ mit einer Inzidenzrelation („$\in$"), das folgende Eigenschaften besitzt:

(M I) *Drei Punkte gehören mindestens einem Kreis an. Enthalten drei verschiedene Kreise zwei gemeinsame Punkte, so schneiden sie sich zu je zweien in den gleichen Punkten.*

(M II) *(Berührsatz) Liegt A auf a und liegt B nicht auf a, so geht durch A, B genau ein Kreis, der mit a nur A gemeinsam hat (a „berührt").*

(M III) *Es gibt vier nicht zugleich auf einem Kreis gelegene Punkte. Jeder Kreis enthält mindestens einen Punkt.*

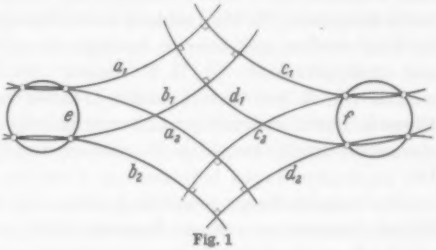


Fig. 1

Dieser Begriff erwuchs aus einer Betrachtung der ebenen Schnitte einer Quadrik $\mathfrak{Q}$ in einem dreidimensionalen projektiven Raum über einem Schiefkörper $\mathfrak{K}$ (dargestellt durch $xA\bar{x}' = 0$, wo $u \to \bar{u}$ ein involutorischer Antiautomorphismus von $\mathfrak{K}$ und $\bar{A}' = A$ ist). Dabei soll $\mathfrak{Q}$ nicht ausgeartet, nicht nullteilig sein und folgende Eigenschaft besitzen: Die Polarebene eines Punktes P von $\mathfrak{Q}$ trifft stets $\mathfrak{Q}$ nur in P und enthält alle Geraden, die $\mathfrak{Q}$ genau in P schneiden. (Im Falle Char. $\mathfrak{K} \neq 2$ ist dies äquivalent mit der Forderung, daß $\mathfrak{Q}$ keine ganze Gerade enthält; siehe BENZ [2].) $\mathfrak{Q}$ heiße dann *schwach konvex*. Das System der nichtleeren ebenen Schnitte von $\mathfrak{Q}$ bezeichnen wir mit $\mathfrak{M}(\mathfrak{K}, \mathfrak{Q})$. Es bildet, wie im Falle Char. $\mathfrak{K} \neq 2$ W. BENZ [2] gezeigt hat, eine Möbiusebene. (In dieser können zwei verschiedene Kreise unendlich viele Schnittpunkte besitzen.)

In der vorliegenden Arbeit werden die Ebenen $\mathfrak{M}(\mathfrak{K}, \mathfrak{Q})$ unter den Möbiusebenen gekennzeichnet[1] [2]), und zwar mit Hilfe eines Schließungssatzes für

[1]) In dem Fall, daß $\mathfrak{Q}$ nicht ausgeartet ist und eine ganze Gerade enthält, liefert die Arbeit [3] des Verf. eine einfache Kennzeichnung des Systems der ebenen Schnitte von $\mathfrak{Q}$ (Ordnet man — was geometrisch unbefriedigend ist — auch bei schwach konvexem und nullteiligem nichtausgeartetem $\mathfrak{Q}$ jeder Ebene einen — evtl. „komplexen" — Schnittkreis mit $\mathfrak{Q}$ zu, dann liefert [3] eine entsprechende Kennzeichnung bei beliebiger nichtausgearteter Quadrik.)

[2]) Zu einer Anregung durch R. BAER vgl. Fußnote [7]).

Inzidenz und Orthogonalität. Dabei verwenden wir einen etwas anderen Orthogonalitätsbegriff als den von W. BENZ [2] untersuchten.

Ferner kennzeichnen wir eine noch allgemeinere Klasse von Möbiusebenen, nämlich diejenigen Ebenen $\mathfrak{M}(\mathfrak{K}, \mathfrak{S})$, die aus den ebenen Schnitten einer beliebigen geeigneten Punktmenge $\mathfrak{S}$ eines projektiven Raumes über einem beliebigen Schiefkörper $\mathfrak{K}$ bestehen (Abschnitt 7). $\mathfrak{S}$ stellt eine Verallgemeinerung der streng konvexen Fläche des R_3 mit überall eindeutiger Stützebene dar (vgl. dazu die Arbeit [4] des Verf.).

Als wesentliches Hilfsmittel beim Aufbau eines projektiven Raumes mit Hilfe der Kreise benutzen wir einen Satz über „projektive Gefüge" (vgl. die Arbeit [5] des Verf.).

Unsere Betrachtungen bilden zugleich einen Ansatz zur Begründung einer räumlich-hyperbolischen Geometrie über Schiefkörpern.

Die Ebenen $\mathfrak{M}(\mathfrak{K}, \mathfrak{Q})$ und $\mathfrak{M}(\mathfrak{K}, \mathfrak{S})$ genügen dem Büschelsatz (siehe Abschnitt 3; vgl. Abb. 12), im allgemeinen jedoch nicht dem Satz von MIQUEL. (Die Gültigkeit des letzteren führt zu kommutativen Körpern $\mathfrak{K}$; vgl. B. L. VAN DER WAERDEN - L. J. SMID [9].) Die schon von VAN DER WAERDEN-SMID und anderen bemerkte Analogie zwischen Büschelsatz, Miquelschem Satz einerseits und DESARGUES, PAPPUS andererseits kann durch den hier betrachteten Schließungssatz (S) für Inzidenz und Orthogonalität insofern als ergänzt betrachtet werden, als dieser in Analogie zu einem Schließungssatz für Inzidenz und Orthogonalität von H. NAUMANN - K. REIDEMEISTER [7] bzw. einem solchen von K. SCHÜTTE [8] steht. Letztere kennzeichnen die affin-metrischen Ebenen. Durch zweimalige Anwendung folgt aus ihnen sofort der Desargues, ebenso wie durch zweimalige Anwendung von (S) sofort der Büschelsatz folgt.— Die Analogien tragen indessen nur formalen, keinen inhaltlichen Charakter.

Die verschiedenen Schließungssätze, die wir in den Beweisen benutzen, führen einerseits zu weiteren Kennzeichnungen der Ebenen $\mathfrak{M}(\mathfrak{K}, \mathfrak{Q})$ (siehe Abschnitt 8), andererseits dürften sie, ebenso wie (S), auch elementargeometrisch interessant sein. Offenbar hat man sie bisher nicht beachtet.

1. (MO)-Ebenen und Schließungssatz

Das Schnittgebilde zweier verschiedener Kreise a, b wird, sofern es mindestens zwei Punkte enthält, *Fährte* genannt. Fährten bezeichnen wir durch $\varphi, \psi, \ldots$ oder einfach durch $a \cap b$ usw. Eine Möbiusebene heißt eine (F)-Ebene, wenn in ihr zusätzlich gilt [2]:

(F) *Berührt k einzeln a_1, a_2, a_3 und ist $a_1 \cap a_2 = a_2 \cap a_3 = a_1 \cap a_3 = \{P\}$, so gibt es mindestens eine Fährte φ mit $k \cap a_i \subset \varphi$ $(i = 1, 2, 3)$.*

Eine (F)-Ebene werde *Möbiusebene mit Orthogonalität* oder kurz (MO)-*Ebene* genannt, wenn für sie eine Orthogonalitätsrelation „$\perp$" mit folgenden Eigenschaften gegeben ist:

(O 1) *Aus $a \perp b$ folgt $b \perp a$.*

(O 2) *Ist $a \perp b$ und $a \cap b \neq \Theta$, so ist $a \cap b$ eine Fährte.*

(O 3) *Ist $a \perp b$, c und $b \cap c = b \cap d$ eine Fährte, so ist $a \perp d$.*

(O 4) *Zu $A \in a$, $B \neq A$ gibt es genau ein b mit $A, B \in b$; $a \perp b$.*

Im 2. Abschnitt beschäftigen wir uns allgemein mit (MO)-Ebenen. Erst später ziehen wir unseren Schließungssatz heran, der hier schon formuliert sei:

Schließungssatz (S). *Die Kreise* a_1, a_2, b_1, b_2, e *bzw.* c_1, c_2, d_1, d_2, f *seien je untereinander verschieden. Es gelte* $|a_1 \cap a_2| \geqq 2$[3]), $|b_1 \cap b_2| \geqq 1$, $|c_1 \cap c_2| \geqq 2$, $|d_1 \cap d_2| \geqq 1$, $a_1 \cap a_2 \neq b_1 \cap b_2$, $(c_1 \cap c_2) \cap (d_1 \cap d_2) = 0$. *Sind dann die Beziehungen*

$$a_i, b_i \perp c_i, d_i \qquad (i = 1, 2)$$

$$a_1 \cap a_2 = a_1 \cap e$$

$$b_1 \cap b_2 = b_1 \cap e$$

$$c_1 \cap c_2 = c_1 \cap f$$

$$d_1 \cap d_2 \cap f \neq 0$$

erfüllt, dann gilt auch

$$d_1 \cap d_2 = d_1 \cap f \,.$$ (Fig. 1)[4]).

Um eine einprägsame Formulierung von (S) zu gewinnen, nennen wir ein System von vier verschiedenen Kreisen a, b, c, d mit $a, b \perp c, d$ ein *Kreisrechteck*. a, b bzw. c, d heißen *gegenüberliegende Seiten* des Kreisrechtecks. Ferner heiße ein Paar von Fähnen *zyklisch*, wenn die Fähnen gemeinsam einem Kreis angehören. Ist $a \cap b = \{P\}$, so bezeichnen wir das System bestehend aus P, a, b und allen Kreisen x mit $a \cap x = b \cap x = \{P\}$ als *Berührfähne*. Der Einfachheit halber sagen wir gelegentlich: die Berührfähne *liegt auf* den Kreisen a, b, x. Eine Fähne und eine Berührfähne bzw. zwei Berührfähnen bilden ein *zyklisches* Paar, wenn sie gemeinsam auf dem Kreis liegen. Folgende Aussage ist dann zu (S) äquivalent:

(S′) *Zwei Kreisrechtecke seien so gelegen, daß je zwei gegenüberliegende Seiten des einen zwei gegenüberliegende Seiten des anderen in verschiedenen Fähnen schneiden. Dabei darf jeweils eine Berührfähne an die Stelle einer der beiden Fähnen treten. Ist dann das eine Paar zyklisch, so auch das andere.*

Um die Äquivalenz von (S) und (S′) nachzuweisen, braucht man sich im wesentlichen nur zu überlegen, daß es keine zu sich selbst orthogonalen (isotropen) Kreise gibt und ein Kreis nicht nur aus einer Fähne besteht. Letzteres folgt nach Benz [2], Satz (4) (siehe nächsten Abschnitt), ersteres ist wegen (O 2) eine Folge daraus (bei $a \perp a$ wäre $a \cap a = a$ eine Fähne). Der in (S) weggelassene Fall $(c_1 \cap c_2) \cap (d_1 \cap d_2) \neq 0$ erledigt sich nach (M I) und Benz [2], Satz (3) (siehe nächsten Abschnitt).

Ziel unserer Betrachtungen ist die folgende Aussage:

Hauptsatz. *Die* (MO)*-Ebenen, die* (S) *bzw.* (S′) *erfüllen, sind genau die Möbiusebenen* $\mathfrak{M}(\mathfrak{K}, \mathfrak{Q})$.

Die Möbiusebenen $\mathfrak{M}(\mathfrak{K}, \mathfrak{Q})$ sind (MO)-Ebenen, sofern man stets dann $a \perp b$ setzt, wenn die durch a, b bestimmten Ebenen bezüglich $\mathfrak{Q}$ konjugiert

[3]) $|\mathfrak{M}|$ bedeutet die Mächtigkeit der Menge $\mathfrak{M}$.

[4]) Man kann (S) auch so aussprechen, daß die Fähnen $a_1 \cap a_2$ usw. durch Punktepaare repräsentiert werden. Es ergibt sich dann ein Schließungssatz für 8 Punkte und 10 Kreise.

sind[5]). Die Eigenschaften (O 1)—(O 4) ergeben sich aus entsprechenden Sätzen über konjugierte Ebenen. (M I)—(M III) folgen, wie wir in Abschnitt 7 sehen werden, sofort aus der schwachen Konvexität von $\mathfrak{Q}$ (mit der in der Einleitung angegebenen Definition der schwachen Konvexität). Der Nachweis von (F) läßt sich aus W. BENZ [2], S. 246 übernehmen, weil dort nur (M II) und eine Eigenschaft der schwachen Konvexität benutzt werden. Auch (S) sieht man ohne Schwierigkeit ein:

Sind x, y zwei verschiedene Kreise, so bezeichnen wir für den Augenblick mit (x, y) die Gerade, in der sich die durch x bzw. y bestimmten Ebenen des betrachteten projektiven Raumes schneiden. Die Voraussetzungen von (S) besagen dann, daß die Geraden (a_1, b_1) und (c_1, d_1) sowie die Geraden (a_2, b_2) und (c_2, d_2) bezüglich $\mathfrak{Q}$ konjugiert sind und daß sich (a_1, a_2), (b_1, b_2), also auch (a_1, b_1), (a_2, b_2) schneiden. Dann schneiden sich aber nach bekannten Sätzen über Polarität auch (c_1, d_1), (c_2, d_2) und somit (c_1, c_2), (d_1, d_2), d. h. die durch c_1, c_2 und d_1, d_2 bestimmten Fährten bzw. Berührfährten liegen auf einem Kreis.

2. Eigenschaften der (MO)-Ebenen

Für Möbiusebenen hat W. BENZ [2] die folgenden Eigenschaften (1)—(6) nachgewiesen:

(1) *Jeder Kreis enthält mindestens drei Punkte.*

(2) *Zwei verschiedene Punkte liegen auf genau einer Fährte.*

(3) *Ein Kreis enthält mit zwei Punkten einer Fährte stets die ganze Fährte.*

(4) *Ein Kreis besteht nie aus nur einer Fährte.*

(4a) *Liegt A nicht auf der Fährte φ, so gibt es genau einen Kreis durch A, φ.*

(5) *Zwei Kreise a, b erfüllen stets genau eine der folgenden Bedingungen: $a \cap b = \emptyset$; $|a \cap b| = 1$; $a \cap b$ ist eine Fährte; $a \cap b = a = b$.*

(6) *Ist $|a \cap \varphi| = 1$, so geht durch φ genau ein Kreis b, der a berührt.*

In (MO)-Ebenen gelten die folgenden Eigenschaften (7), (8), deren Beweise man von BENZ [2] wörtlich übernehmen kann, wenn man dort (O I), (O II), (O III) bzw. durch (O 1), (O 2), (O 4) ersetzt:

(7) *Ist für einen Kreis a und eine Fährte φ $a \cap \varphi \neq \emptyset$, so geht durch φ genau ein Kreis b mit $b \perp a$.*

(8) *Ist $b \perp a$ und $c \cap b = \{P\} \subset a$, so folgt $c \perp a$.*

Wir zeigen nun:

(9) *Sei in einer (MO)-Ebene $P \in b, c$; $b \neq c$; $b, c \perp a$. Dann ist entweder $b \cap c = \{P\}$ oder es ist $b \cap c$ eine Fährte, je nachdem, ob $P \in a$ oder $P \notin a$ gilt. Ist im Falle $P \notin a$ ein weiterer Kreis d mit $P \in d$; $d \perp a$ gegeben, so folgt $b \cap c \subset d$.*

Beweis. Ist $P \in a$, so folgt aus $Q \in b, c$; $Q \neq P$ ein Widerspruch zu (O 4). Sei also $P \notin a$.

Zu jedem $X \in a$ gibt es nach (O 4) einen Kreis x mit $P, X \in x$; $x \perp a$. Angenommen, diese Kreise x berühren sich zu je zweien, dann legen wir durch jedes X den Kreis x' mit $P \in x'$; $x' \perp x$. Wegen (8) ist dann jedes x' zu jedem x

[5]) In BENZ [2] wird zusätzlich $a \cap b \neq \emptyset$ verlangt. Dies bedeutet den Verzicht auf (O 3).

orthogonal. Nach dem ersten Teil des Beweises berühren sich je zwei der Kreise x' in P; ferner haben wir $a \cap x' = \{X\}$ für jedes x'. Durch mehrfache Anwendung von (F) folgt somit, daß alle Punkte $X \in a$ einer Fährte angehören, im Widerspruch zu (4).

Es gibt also zwei Kreise x_1, x_2 unter den x, für die $x_1 \cap x_2$ eine Fährte ist. Sei b jetzt irgendein Kreis mit $P \in b$; $b \perp a$ und nehmen wir an, b enthält die Fährte $x_1 \cap x_2$ nicht. Da wegen (6) x_1 und x_2 nicht beide den Kreis b berühren, können wir etwa voraussetzen, daß $x_2 \cap b$ eine Fährte ist.

Sei $X_1 \in a \cap x_1$. Dann liegt nach (O 4) X_1 weder auf x_2 noch auf b. Also bestimmen $x_2 \cap b$ und X_1 nach (4a) einen Kreis b'. Dieser ist nach (O 3) zu a orthogonal. Wäre $b' = x_1$, so folgte entgegen Annahme $x_1 \cap x_2 \subset b$. Somit gehen durch P, X_1 zwei Orthogonalkreise von a (nämlich b' und x_1), entgegen (O 4). Es gilt also $x_1 \cap x_2 \subset b$.

Genauso folgt $x_1 \cap x_2 \subset c$ und $x_1 \cap x_2 \subset d$, wenn c, d die im Satz genannten Kreise sind. Bei $c \neq b$ folgt wegen (M I): $x_1 \cap x_2 = b \cap c$, so daß $b \cap c$ eine Fährte ist.

(10) *Zu P, a, b mit $P \notin a$ gibt es stets ein c, so daß $P \in c$; $c \perp a$, b ist. Gilt $P \in b$, so ist c eindeutig bestimmt.*

Beweis. Ist $a = b$, so folgt die Behauptung aus (O 4). Sei also $a \neq b$. Wir finden zunächst auf Grund von (O 4) zwei verschiedene Kreise p, q mit $P \in p, q$; $p, q \perp a$. Nach (9) ist $p \cap q$ eine Fährte. Bei $P \in b$ gibt es nach (7) genau einen Kreis c mit $p \cap q \subset c$; $c \perp b$. Nach (O 3) ist c auch zu a orthogonal, also der gesuchte, eindeutig bestimmte Kreis.

Ist $P \notin b$, dann suchen wir wieder auf Grund von (O 4) zwei Kreise r, s mit $P \in r$, s und r, $s \perp b$. Nach (4a) und (3) folgt die Existenz eines Kreises c mit $p \cap q \subset c$; $r \cap s \subset c$. Wegen (O 3) hat man $c \perp a$, b.

(11) *Sind b, b', b'' verschiedene Kreise mit $b, b', b'' \perp a, a'$; $a \neq a'$, dann folgt $b \cap b' = b \cap b'' = b' \cap b''$.*

Beweis. Sind je zwei der Kreise b, b', b'' zueinander punktfremd, so ist nichts zu beweisen. Sei also $P \in b \cap b'$. Ist $P \in b''$, dann folgt die Behauptung aus (9). Andernfalls sei c ein Kreis mit $b \cap b' = b \cap c$ und $c \cap b'' \neq \emptyset$. Die durch b, b' bzw. c, b'' bestimmten Fährten bzw. Berührfährten enthalten außer c keinen gemeinsamen Kreis. Man lege durch jede von ihnen und einen Punkt A mit $A \in a$; $A \notin c$ je einen Kreis d bzw. d'. Dann ist $d \neq d'$ und nach (O 3) bzw. (8) $d, d' \perp a, a'$ entgegen (10) (im Falle $A \notin a'$) oder (O 4) (im Falle $A \in a'$ und $|a \cap a'| \geqq 2$) oder (9) (im Falle $a \cap a' = \{A\}$ wegen $A \notin c$; $c \perp a, a'$).

(12) *Sind die Kreise p_1, p_2, q_1, q_2 alle verschieden, ist $a \perp p_i, q_i$ $(i = 1, 2)$; $p_1 \cap p_2 \neq \emptyset$; $q_1 \cap q_2 \neq \emptyset$; $p_1 \cap (q_1 \cap q_2) = q_1 \cap (p_1 \cap p_2) = \emptyset$, dann gibt es ein b mit $p_1 \cap p_2 = p_1 \cap b$; $q_1 \cap q_2 = q_1 \cap b$.*

Beweis. Sei $P \in p_1 \cap p_2$; $Q \in q_1 \cap q_2$. Ist b ein Kreis mit $Q \in b$; $p_1 \cap p_2 = p_1 \cap b$ (Existenz im Falle $p_1 \cap p_2 = \{P\}$ nach (M II), sonst nach (4a)), so ergibt (O 3) bzw. (8), daß $a \perp b$ gilt. Liegt Q auf a, so ist nach (9) $q_1 \cap q_2 = q_1 \cap b = \{Q\}$. Liegt Q nicht auf a, so folgt $q_1 \cap q_2 = q_1 \cap b$ nach dem zweiten Teil von (9).

(13) *Zu P, Q, a gibt es stets ein b mit $P, Q \in b$ und $b \perp a$.*

Beweis. Liegen P, Q beide auf a, so folgt die Behauptung aus (O 4). Wir nehmen daher etwa $P \notin a$ an. Mit Hilfe von (O 4) finden wir verschiedene Kreise p_1, p_2 mit $P \in p_1, p_2$; $p_1, p_2 \perp a$. Nach (9) ist $p_1 \cap p_2$ eine Fährte. Wegen (4a) gibt es einen Kreis durch $p_1 \cap p_2$ und Q; dieser ist nach (O 3) zu a orthogonal.

(14) *Gilt* $a \cap b = \{P\}$ *und* $P \notin c$, *so gibt es ein* d *mit* $d \perp a, b, c$.

Beweis. Nach (10) gibt es ein d mit $P \in d$; $d \perp a, c$. Nach (8) folgt $d \perp b$.

(15) *Auf jedem Kreis liegen mindestens vier Punkte, die zu je dreien nicht gemeinsam einer Fährte angehören.*

Beweis. Sei a irgendein Kreis. Nach (1), (3), (4) gibt es einen Kreis b und einen Punkt A mit $A \in a$; $A \notin b$, so daß $a \cap b$ eine Fährte ist. Ist c der (nach (10) existierende) Kreis mit $A \in c$; $c \perp a, b$, so folgt nach (O 2) und (9), daß $a \cap c$ eine Fährte ist, die keinen Punkt von $a \cap b$ enthält. Auf $a \cap b$ und $a \cap c$ findet man somit Punkte der gewünschten Art.

Satz 1. *Eine (F)-Ebene gestattet höchstens eine Orthogonalitätsrelation der Stärke* (O 1)—(O 4).

Beweis. Seien zwei Orthogonalitätsrelationen „$\perp$“ und „$\top$“ in der (F)-Ebene mit jeweils den definierenden Eigenschaften (O 1)—(O 4) gegeben. Nehmen wir sie als verschieden an, so gibt es etwa a, b mit $a \perp b$; $a \not\top b$. Ist $a \cap b \neq \emptyset$, also nach (O 2) $a \cap b$ eine Fährte, so können wir wörtlich den Beweis für den entsprechenden Satz bei BENZ [2], S. 242 übernehmen. Im Falle $a \cap b = \emptyset$ brauchen wir dann nur noch zu zeigen, daß es einen Kreis b' gibt mit $a \perp b'$, $a \not\top b'$, $a \cap b' \neq \emptyset$.

Sei in der Tat P ein Punkt von b, seien Q_1, Q_2 Punkte von a und seien q_1, q_2 die Kreise mit $P, Q_1 \in q_1$; $P, Q_2 \in q_2$; $q_i \perp a$ $(i = 1, 2)$. Wir können Q_1, Q_2 so wählen, daß $q_1 \neq q_2$ ist. Nach (9) ist $q_1 \cap q_2$ eine Fährte (aus $a \cap b = \emptyset$ folgt $P \notin a$) und liegt auf b. Wäre sowohl $q_1 \top a$ wie $q_2 \top a$, so hätte man nach (O 3) $b \top a$. Wir können daher $q_1 \not\top a$ annehmen und $b' = q_1$ setzen.

3. Der Büschelsatz

Gegeben sei eine (MO)-Ebene, in der (S) bzw. (S′) gilt.

Büschelsatz. *Liegen vier Paare lauter verschiedener Punkte in fünf Fällen zu je zweien auf einem Kreis und sind diese Kreise alle verschieden, dann liegen auch im sechsten Fall die Paare auf einem Kreis.*

Beweis. Wir bezeichnen die 5 gegebenen Kreise so mit a_1, a_2, b_1, b_2, e, daß $a_1 \cap a_2 = a_1 \cap e$, $b_1 \cap b_2 = b_1 \cap e$ gilt und die vier Punktepaare bzw. auf folgenden Fährten liegen (und sie wegen (3) bestimmen): $a_1 \cap a_2$, $a_1 \cap b_2$, $a_2 \cap b_1$, $b_1 \cap b_2$. Wir können annehmen, daß diese Fährten zu je zweien punktfremd sind (sonst ist der Satz trivial). Dann ist die Existenz eines Kreises f nachzuweisen, so daß $a_1 \cap b_2 = a_1 \cap f$ und $b_1 \cap a_2 = b_1 \cap f$ gilt.

P sei ein Punkt mit $P \notin a_1 \cap b_1$ und $P \notin a_2 \cap b_2$ (etwa $P \in a_1 \cap a_2$) und c_1, c_2 seien Kreise durch P mit $c_i \perp a_i, b_i$ $(i = 1, 2)$; sie existieren nach (10) (Abb. 2). Wir können erreichen (wieder durch $P \in a_1 \cap a_2$), daß $c_1 \cap c_2$ eine Fährte ist. Sei Q ein weiterer Punkt mit $Q \notin a_1 \cap b_1$; $Q \notin a_2 \cap b_2$ und $Q \notin c_1$;

$Q \notin c_2$. (Man wähle etwa $P \in a_1 \cap a_2$. Enthielten c_1, c_2 sämtliche Fährten $a_1 \cap b_2$, $a_2 \cap b_1$, $b_1 \cap b_2$, dann fielen c_1, c_2 entweder mit den Kreisen a_1, b_1 oder mit a_2, b_2 oder mit e, f zusammen, wo f der gesuchte Kreis ist. Im letzten Falle wären wir fertig, in den beiden ersten Fällen wäre einer der beiden Kreise c_1, c_2 isotrop. Also gibt es in einer der Fährten $a_1 \cap b_2$, $a_2 \cap b_1$, $b_1 \cap b_2$ einen Punkt Q der gewünschten Art.) d_i seien Kreise durch Q mit $d_i \perp a_i, b_i$ $(i = 1, 2)$. Sie sind beide von c_1, c_2 verschieden. Gilt $d_1 = d_2$, so folgt der Büschelsatz aus (12). Sei also $d_1 \neq d_2$.

Man lege durch $c_1 \cap c_2$ und Q einen Kreis f'. Die Kreise c_1, c_2, d_1, d_2, f' sind dann alle verschieden. Ist $(c_1 \cap c_2) \cap (d_1 \cap d_2) = \emptyset$, so folgt nach (S): $d_1 \cap d_2 = d_1 \cap f'$. Dasselbe gilt bei $(c_1 \cap c_2) \cap (d_1 \cap d_2) \neq \emptyset$ wegen (M I) und (3).

Man lege jetzt durch $a_1 \cap b_2$ und einen Punkt von $a_2 \cap b_1$ einen Kreis f. Ersetzt man dann in (S) die Kreise $a_1, a_2, b_1, b_2, e, c_1, c_2, d_1, d_2, f$ der Reihe nach durch $c_1, c_2, d_1, d_2, f', a_1, b_2, b_1, a_2, f$, so folgt nach (S) $b_1 \cap a_2 = b_1 \cap f$.

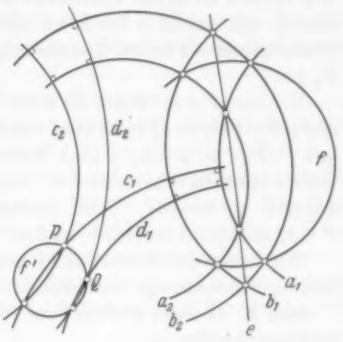


Fig. 2

Erweiterter Büschelsatz. *Sind von vier zu je zweien punktfremden Fährten, die zum Teil oder alle durch Berührfährten ersetzt werden dürfen, fünf Paare zyklisch und sind die so gegebenen fünf Kreise alle verschieden, dann ist auch das sechste Paar zyklisch.*

Beweis. Es ist nur noch der Fall zu behandeln, daß unter den Fährten mindestens eine Berührfährte vorkommt. Sei etwa (in der obigen Bezeichnungsweise) $a_1 \cap b_2 = \{P\}$. Dann liegt P, wie aus den Annahmen folgt, auf keinem der Kreise a_2, b_1, e. Somit existiert nach (10) etwa ein p mit $p \perp a_1, a_2$ und $P \in p$. Nach (8) folgt $p \perp b_2$. Ist $a_1 \cap a_2 = a_1 \cap e = \{Q\}$, so hat man nach (9) $Q \in p$, also wegen (8): $p \perp e$. Ist $a_1 \cap a_2 = a_1 \cap e$ eine Fährte, so folgt $p \perp e$ unmittelbar aus (O 3). Entsprechend schließt man auf $p \perp b_1$. Also besitzen a_1, b_2, a_2, b_1 einen gemeinsamen Orthogonalkreis und die Behauptung folgt aus (12). — Genauso schließt man, wenn sich andere Paare der Kreise a_1, a_2, b_1, b_2 berühren.

4. Behandlung eines Sonderfalles

In den nächsten Abschnitten werden wir folgende Zusatzannahme benötigen:

(Z) Sind P, a beliebig mit $P \in a$, so gibt es vier verschiedene Fährten φ_i mit $P \in \varphi_i \subset a$ $(i = 1, \ldots, 4)$.

Wir untersuchen in diesem Abschnitt den Fall, daß (Z) nicht gilt. Dabei beweisen wir einen allgemeineren Satz (Satz 2), der von eigenem Interesse sein dürfte.

(16) *Gibt es in einer Möbiusebene ein Paar P_0, a_0 mit $P_0 \in a_0$ und eine Menge von n verschiedenen Fährten $\{\varphi_1^0, \ldots, \varphi_n^0\}$ mit $P_0 \in \varphi_\varrho^0 \subset a_0$ $(\varrho = 1, \ldots, n)$,*

derart, daß jeder Punkt $\neq P_0$ von a_0 genau einer der Fährten φ_ϱ^0 angehört, dann gibt es zu jedem Paar P, a mit $P \in a$ eine Menge von n verschiedenen Fährten $\{\varphi_1, \ldots, \varphi_n\}$ mit $P \in \varphi_\varrho \subset a$, so daß jeder Punkt $\neq P$ von a auf genau einem φ_ϱ liegt.

Beweis. Die Gesamtheit der Kreise durch P_0 bildet zusammen mit den Fährten, die P_0 enthalten (als Punkten), eine affine Ebene (BENZ [2], S. 241). Wir nennen sie *affine Unterebene* der Möbiusebene. a_0 repräsentiert dabei eine Gerade mit genau n Punkten. Also liegen auf jeder Geraden dieser affinen Ebene genau n Punkte. Daraus folgt die Behauptung für alle Paare P_0, a mit $P_0 \in a$.

Sei nun $P \notin a_0$. Dann ist auch $P \notin \varphi_\varrho^0$ $(\varrho = 1, \ldots, n)$. Daher bestimmt P mit jeder Fährte φ_ϱ^0 nach (4a) einen Kreis. Ferner gibt es genau einen Kreis p mit $P, P_0 \in p$; $p \cap a_0 = \{P_0\}$. Somit gehen durch P, P_0 genau $n+1$ Kreise. Genau einer davon berührt a [wegen (M II)]. Die n übrigen schneiden a in Fährten φ_ϱ $(\varrho = 1, \ldots, n)$. Entsprechende Schlüsse wendet man nun für $P \in a_0$ an. Damit ist (16) bewiesen.

Wir haben im Beweis nicht benutzt, daß n endlich ist. ϱ kann also eine beliebige Indexmenge durchlaufen. Wir erhalten somit folgendes Ergebnis:

Satz 2. *In einer Möbiusebene besitzen sämtliche affinen Unterebenen gleichmächtige Geraden.*

Jedoch brauchen zwei affine Unterebenen nicht isomorph zu sein (siehe dazu die Arbeit [4] des Verf.).

Im folgenden benötigen wir (16) nur für $n = 3$.

(17) *Gilt in einer Möbiusebene die Negation von* (Z), *so liegen drei verschiedene Punkte stets auf genau einem Kreis und enthalten die Kreise entweder alle genau drei oder alle genau vier Punkte.*

Beweis. Wir bezeichnen im folgenden mit $(X\,Y)$ die durch X, Y bei $X \neq Y$ bestimmte Fährte. — Angenommen, es gibt eine Fährte φ, die drei verschiedene Punkte A, B, C enthält. a sei ein Kreis durch φ und P ein Punkt mit $P \in a$; $P \notin \varphi$ [Existenz nach (4)]. Die Fährten (PA), (PB), (PC) sind dann verschieden. Nach (16) und der Negation von (Z) gibt es somit genau drei verschiedene Fährten $\varphi_1, \varphi_2, \varphi_3$ mit $A \in \varphi_i \subset a$ $(i = 1, 2, 3)$. Eine davon fällt mit φ zusammen, eine zweite enthält P. In der dritten existiert ein Q mit $Q \notin \varphi$; $Q \notin (AP)$. Gilt auch $Q \notin (BP)$, $Q \notin (CP)$, dann sind die Fährten (AP), (BP), (CP), (QP) alle verschieden, was wegen (16) der Negation von (Z) widerspricht. Sei also etwa $Q \in (BP)$. Dann finden wir wie eben ein $R \in a$ mit $R \notin \varphi$; $R \notin (BP)$. Im Falle $R \notin (AP)$, (CP) erhalten wir wieder einen Widerspruch. Ist etwa $R \in (AP)$, so sind die Fährten (AR), (BR), (CR), (QR) verschieden, was ebenfalls nicht geht.

Somit enthält jede Fährte genau zwei Punkte. Nach Satz 2 folgt hieraus, daß sämtliche Kreise gleich viele Punkte besitzen, und zwar höchstens vier. Nach (4) enthält jeder Kreis mindestens drei Punkte. Daraus folgt die Behauptung.

Gilt die Negation von (Z) in einer (MO)-Ebene, dann liegen wegen (15) auf jedem Kreis genau vier Punkte. Setzt man (S) voraus, so folgt nach Ab-

schnitt 3 die Gültigkeit des erweiterten Büschelsatzes. Nach einem unveröffentlichten Ergebnis von A. J. HOFFMAN[6]) kann man eine endliche Möbiusebene, in der durch drei Punkte stets höchstens ein Kreis geht und auf jedem Kreis eine gerade Anzahl von Punkten liegt sowie der erweiterte Büschelsatz gilt, als das System der ebenen Schnitte einer nichtausgearteten Quadrik eines projektiven Raumes auffassen. Dabei enthält die Quadrik keine ganze Gerade. Somit hat man:

Satz 3. *Der Hauptsatz (siehe Abschnitt 1) gilt bei Annahme der Negation von* (Z). *Ist* $\mathfrak{M}(\mathfrak{K}, \Omega)$ *die betreffende Möbiusebene, dann ist* $\mathfrak{K}$ *der Restklassenkörper* mod 3.

Für spätere Zwecke beweisen wir noch zwei Hilfssätze:

(18) *In einer Möbiusebene gibt es unter der Annahme* (Z) *zu einem Paar* P, a *mit* $P \in a$ *stets* 5 *verschiedene Fährten* φ_i *mit* $P \in \varphi_i \subset a$ $(i = 1, \ldots, 5)$.

Beweis. Nach (16) können wir annehmen, daß auf a 5 verschiedene Punkte P, Q, R, S, T liegen, die zu je dreien keiner Fährte angehören. Wir legen durch P, Q einen Kreis $b \neq a$. Ferner seien r, s, t die Kreise mit $R \in r$; $S \in s$; $T \in t$; $r, s, t \perp a, b$. Von diesen sind mindestens zwei verschieden, etwa r, s. Dann sind die Fährten $r \cap a$ und $s \cap a$ punktfremd [nach (10)]. Man findet in ihnen ein $U \neq R, S, T$, so daß die Fährten (PQ), (PR), (PS), (PT), (PU) alle verschieden sind.

(19) *Seien* $a_1, a_2, a_3, \bar{a}_1, \bar{a}_2, \bar{a}_3$ *sechs Kreise einer* (MO)*-Ebene, so daß* $a_1 \cap a_2$, $a_1 \cap a_3$, $a_2 \cap a_3$, $\bar{a}_1 \cap \bar{a}_2 = \bar{a}_1 \cap \bar{a}_3 = \bar{a}_2 \cap \bar{a}_3$ *Fährten sind und* $a_1 \cap a_2 \cap a_3 = \{A\}$ *sowie* $A \in \bar{a}_1 \cap \bar{a}_2$ *gilt (vgl. solche Kreise in Abb.* 3*). Dann gibt es unter der Annahme* (Z) *einen Punkt* Q*, der auf keinem dieser sechs Kreise liegt.*

Beweis. Einer der Kreise a_1, a_2, a_3, etwa a_1, ist von $\bar{a}_1, \bar{a}_2, \bar{a}_3$ verschieden. Geht jeder der Kreise $\bar{a}_1, \bar{a}_2, \bar{a}_3$ durch eine andere der Fährten $a_1 \cap a_2$, $a_1 \cap a_3$, $a_2 \cap a_3$ hindurch, so findet man nach (18) auf einem der a_i, etwa auf a_1 eine Fährte φ, die auf keinem der Kreise $a_2, a_3, \bar{a}_1, \bar{a}_2, \bar{a}_3$ liegt. Der durch $\varphi, a_2 \cap a_3$ bestimmte Kreis enthält dann, wieder nach (18), eine Fährte, die auf keinem der Kreise $a_1, a_2, a_3, \bar{a}_1, \bar{a}_2, \bar{a}_3$ gelegen ist. In dieser gibt es einen Punkt der gewünschten Art. — Liegt etwa $a_1 \cap a_2$ auf keinem der Kreise $\bar{a}_1, \bar{a}_2, \bar{a}_3$, dann legt man durch $a_1 \cap a_2$ und $\bar{a}_1 \cap \bar{a}_2$ einen Kreis und wendet denselben Schluß wie eben an.

5. Die Dreiecklotsätze

Bei den folgenden Betrachtungen gehen wir von der schon erwähnten Tatsache aus, daß in einer Möbiusebene die Fährten und Kreise durch einen beliebigen festen Punkt A eine affine Ebene bilden (W. BENZ [2]). Wir denken uns diese Ebene zu einer projektiven Ebene erweitert, indem wir die Berührfährten in A *uneigentliche Punkte* und ihre Gesamtheit *uneigentliche Gerade* nennen. Nicht uneigentliche Punkte bzw. Geraden heißen *eigentlich*. Es ist klar, wie sich die Inzidenz in dieser projektiven Ebene in den Begriffen der Möbiusebene ausdrückt. Die so entstandene Ebene nennen wir *projektive*

[6]) Mitgeteilt in einem Vortrag am Mathematischen Institut der Universität Mainz.

Unterebene $\Pi(A)$ der Möbiusebene. Für die Punkte von $\Pi(A)$ benutzen wir die Buchstaben $P, Q, \ldots$.

Wir definieren nun folgenden Begriff, der zwei Ebenen $\Pi(A)$ und $\Pi(B)$ mit $A \neq B$ zueinander in Beziehung setzt[7]: Ein Punkt P von $\Pi(A)$ und ein Punkt P' von $\Pi(B)$ heißen *$A\,B$-perspektiv*, wenn 1. A nicht auf der gegebenenfalls zu P gehörigen Fährte liegt sowie 2. es einen Kreis durch A, B gibt, auf dem die zu P, P' gehörigen Fährten bzw. Berührfährten liegen.

Die Figur, bestehend aus drei nichtkollinearen Punkten P, Q, R von $\Pi(A)$ und ihren Verbindungsgeraden, nennen wir *A-Dreieck*. Die Figur aus vier zu je dreien nichtkollinearen Punkten P, Q, R, S von $\Pi(A)$, den Geraden PQ, QR, RS, SP sowie den Schnittpunkten von PQ, RS und PS, RQ heiße *A-Viereck*. Ein A-Dreieck bzw. A-Viereck nennen wir *eigentlich*, wenn alle seine Punkte eigentlich sind. Ein A- und ein B-Dreieck (-Viereck) bezeichnen wir als *$A\,B$-perspektiv*, wenn 1. B auf keiner Seite des A-Dreiecks (A-Vierecks) und A auf keiner Seite des B-Dreiecks (B-Vierecks) liegt sowie 2. die Punkte der beiden Dreiecke (Vierecke) in Paare $A\,B$-perspektiver Punkte aufgeteilt werden können.

Ist a eine Gerade von $\Pi(A)$ und b eine Gerade von $\Pi(B)$ und ist $A \notin b$, so gibt es nach (10) genau einen Kreis $\bar{a}$ durch A mit $\bar{a} \perp a, b$. Wir nennen ihn das *A-Lot* von a, b.

Wir führen noch folgende Sprechweise ein, die sowohl in der Möbiusebene wie in $\Pi(A)$ gedeutet werden kann: Drei Kreise durch A heißen *im Büschel gelegen*, wenn sie gemeinsam eine Fährte oder eine Berührfährte enthalten.

Erster Dreiecklotsatz. *B liege auf keiner Seite eines eigentlichen A-Dreiecks, A auf keiner Seite eines B-Dreiecks mit höchstens einem uneigentlichen Punkt.*

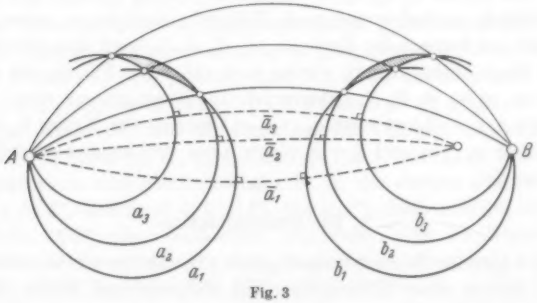


Fig. 3

Lassen sich die Seiten der beiden Dreiecke so einander zuordnen, daß zwei Paare entsprechender Seitenschnittpunkte $A\,B$-perspektiv sind und die A-Lote entsprechender Seiten in einem Büschel liegen, dann sind die Dreiecke $A\,B$-perspektiv (Abb. 3).

[7]) Einen Ausgangspunkt für die vorliegende Arbeit bildete eine Anregung von R. BAER, zwecks Kennzeichnung der Ebenen $\mathfrak{M}(\mathfrak{R}, \Omega)$ von der Gesamtheit der affinen Unterebenen einer Möbiusebene auszugehen.

Beweis. a_1, a_2, a_3 bzw. b_1, b_2, b_3 seien die Kreise des A- bzw. B-Dreiecks. Es sei die Existenz eines Kreises p nachzuweisen, der sowohl mit a_2, a_3 wie mit b_2, b_3 in einem Büschel liegt. Die A-Lote $\bar{a}_i$ ($i = 1, 2, 3$) schneiden sich in einer Fährte, da sich sonst nach (9) die Kreise a_i in A berührten. Fielen zwei der Kreise $\bar{a}_i$ zusammen, dann berührten sich ebenfalls zwei der Kreise a_i, so daß das A-Dreieck nicht eigentlich wäre. Die $\bar{a}_i$ sind also alle verschieden.

Nach (18) existiert ein Punkt C, der auf keinem der Kreise $a_i, \bar{a}_i$ liegt ($i = 1, 2, 3$). Nach (10) gibt es dann Kreise c_i mit $c_i \perp a_i, b_i$; $C \in c_i$ ($i = 1, 2, 3$). Die c_i sind alle von a_1, a_2, a_3 verschieden. Wäre $c_1 = c_2 = c_3$, so lägen nach (9) a_1, a_2, a_3 in einem Büschel entgegen Annahme. Da bei $c_2 = c_3$ die Behauptung nach (12) folgt, können wir somit etwa $c_2 \neq c_3$; $c_1 \neq c_3$ annehmen. $a_1, b_1, \bar{a}_1, c_1$ und $a_3, b_3, \bar{a}_3, c_3$ bilden Kreisrechtecke. Die Seitenpaare a_1, a_3 und b_1, b_3 bilden nach Annahme zyklische Fährten bzw. ein zyklisches Paar Fährte-Berührfährte. Die Seiten $\bar{a}_1, \bar{a}_3$ bilden eine Fährte. Diese ist nach (S′) zyklisch mit der durch c_1, c_3 bestimmten Fährte bzw. Berührfährte. Ganz entsprechend folgt dann umgekehrt, wieder nach (S′), daß der gesuchte Kreis p existiert.

Zweiter Dreiecklotsatz. *Sind ein eigentliches A-Dreieck und ein B-Dreieck mit höchstens einem uneigentlichen Punkt AB-perspektiv, dann liegen die A-Lote entsprechender Seiten in einem Büschel.*

Beweis. Wir führen die gleichen Bezeichnungen wie beim ersten Dreiecklotsatz ein (vgl. Abb. 3). $a_1 \cap a_2$, $a_1 \cap a_3$ und $a_2 \cap a_3$ sind Fährten, da sich sonst zwei der Kreise a_1, a_2, a_3 berührten. Seien etwa die durch b_1, b_2 und b_1, b_3 bestimmten Punkte von $\Pi(B)$ eigentlich. — Wäre sowohl $\bar{a}_3 \perp b_1$ wie $\bar{a}_3 \perp b_2$, dann lägen wegen (9) b_1, b_2, b_3 in einem Büschel entgegen Annahme. Bis auf Umbezeichnung können wir daher $\bar{a}_3 \not\perp b_1$ voraussetzen. Sei q der Kreis durch $a_1 \cap a_2, b_1 \cap b_2$.

Gemäß (7) gibt es ein $\bar{a}_2'$ mit $\bar{a}_1 \cap \bar{a}_2 \subset \bar{a}_2'$ und $\bar{a}_2' \perp a_2$. Man hat $\bar{a}_2' \neq \bar{a}_1, \bar{a}_3$, da sich sonst wieder zwei der Kreise a_1, a_2, a_3 berührten. Sei ferner b_2' der Kreis mit $b_1 \cap b_2 \subset b_2'$ und $b_2' \perp \bar{a}_2'$. Wäre $b_2' = b_1$, so folgte $b_2' \perp \bar{a}_1, \bar{a}_2'$ also nach (O 3): $b_2' = b_1 \perp \bar{a}_3$ entgegen der eben getroffenen Voraussetzung. Hätte man $A \in b_2'$, also $b_2' = q$, dann verliefen durch A die beiden Orthogonalkreise a_2 und b_2' von $\bar{a}_2$. Diese berührten sich also nach (9), so daß einer der Punkte des A-Dreiecks entgegen Annahme uneigentlich wäre. Somit können wir den ersten Dreiecklotsatz anwenden und finden ein p', das mit a_2, a_3 und b_2', b_3 je in einem Büschel liegt. Wegen $B \notin a_2 \cap a_3$ ist aber nach (4a) $p = p'$, also auch $b_2 = b_2'$, $\bar{a}_2 = \bar{a}_2'$. Damit ist der zweite Dreiecklotsatz bewiesen.

6. Der Viereckssatz

Wir beweisen jetzt einen reinen Inzidenzsatz. Zu den Bezeichnungen siehe Abschnitt 5.

Viereckssatz. *A liege auf keiner Seite eines B-Vierecks, B auf keiner Seite eines A-Vierecks. Lassen sich die Seiten der beiden Vierecke so einander zuordnen, daß in fünf Fällen die Schnittpunkte von Paaren entsprechender Seiten AB-perspektiv sind, dann gilt dies auch im sechsten Fall (Abb. 4).*

Beim Viereckssatz handelt es sich im allgemeinen um einen Schließungssatz mit 14 Kreisen. Jedoch kann in jedem der beiden Vierecke eine Seite uneigentlich sein, so daß sich die Zahl der Kreise um 1 oder 2 reduziert. Wir sprechen dann beziehungsweise vom *einfach entarteten* und *zweifach entarteten* Viereckssatz. Ferner können gewisse Kreise durch A, B zusammenfallen. Wir führen das gleich näher aus. Seien $a_1, \ldots, a_4$ die Seiten des A-Vierecks und $b_1, \ldots, b_4$ die Seiten des B-Vierecks. Die durch a_i, a_k $(i < k;\ i = 1, 2, 3;$

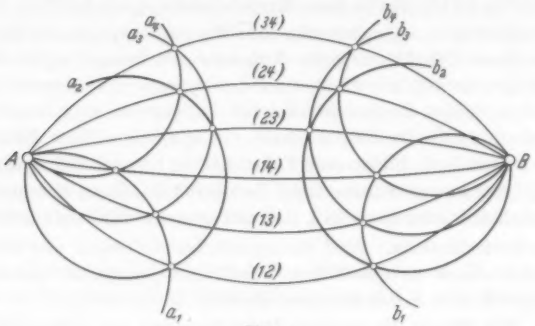


Fig. 4

$k = 2, 3, 4)$ jeweils bestimmte Fährte oder Berührfährte definiert einen Punkt von $\Pi(A)$; wir nennen ihn P_{ik}. Entsprechend definieren wir $Q_{ik} \in \Pi(B)$ mit Hilfe von b_i, b_k. Den Kreis durch A, B, Q_{ik} bezeichnen wir stets mit (ik). Die AB-Perspektivität von P_{ik}, Q_{ik} besagt dann, daß P_{ik} auf (ik) liegt. Aus den Voraussetzungen des Viereckssatzes folgt:

(6.1) Zwei Kreise (ik), (jl) fallen höchstens dann zusammen, wenn die Ziffern i, k, j, l alle verschieden sind.

1. Viereckssatz mit eigentlichen Geraden

Wir bedienen uns, um den Beweis übersichtlich zu gestalten, der folgenden schematischen Darstellung (die bereits den im nächsten Abschnitt vorgenommenen Aufbau eines projektiven Raumes andeutet): Jedem Paar a_j, b_j ordnen wir eine Gerade j eines Schemas zu. Sind P_{ik} und Q_{ik} AB-perspektiv und ist P_{ik} eigentlich, so entspreche dem Paar P_{ik}, Q_{ik} ein Punkt (i, k) des Schemas. Sind P_{ik} und Q_{ik} AB-perspektiv und ist P_{ik} uneigentlich, so nennen wir die Seiten i, k *parallel* (Darstellung im Schema gemäß Abb. 5a). Ist P_{ik} uneigentlich, soll aber erst nachgewiesen werden, daß P_{ik}, Q_{ik} AB-perspektiv sind, dann nennen wir i, k *quasiparallel* und drücken das im Schema gemäß Abb. 5b aus. Durch einen ungefüllten Kreis geben wir auch sonst an, daß ein Schnittpunkt zweier Geraden des Schemas als vorhanden nachgewiesen werden soll. $\bar{i}$ entspreche im Schema dem A-Lot $\bar{a}_i$ von a_i, b_i. Jedem i ist also genau ein *Lot* $\bar{i}$ zugeordnet. Ein Schnittpunkt zweier Lote $\bar{i}, \bar{k}$ besage, daß

a) *parallel* b) *quasiparallel*
Fig. 5

entweder $a_i = a_k$ oder $a_i \cap a_k$ ein eigentlicher Punkt von $\Pi(A)$ (also Fährte) ist und repräsentiere im letzteren Fall diesen Punkt. Zwei Lote schneiden sich danach in höchstens einem Punkt.

Die Dreiecklotsätze drücken sich mit diesen Festsetzungen bzw. folgendermaßen aus:

(a) Haben drei Geraden des Schemas verschiedene Lote, treffen sich diese Lote in einem Punkt und schneiden sich zwei Paare der Geraden, dann schneidet sich auch das dritte Paar (Abb. 6).

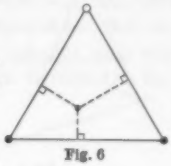


Fig. 6

(b) Schneiden sich im Schema drei verschiedene Geraden zu je zweien und sind die Schnittpunkte verschieden, so treffen sich ihre Lote in einem Punkt.

Ferner ergibt (12) unmittelbar:

(c) Die Lote zweier Geraden fallen genau dann zusammen, wenn die Geraden parallel sind.

Aus (12) und (7) folgt:

(d) Haben die Lote zweier quasiparalleler Geraden einen Schnittpunkt, dann fallen sie zusammen.

Schließlich hat man aus (3) und (4a):

(e) Zwei Punkte (i, k) und (j, l) liegen bei $i \neq j$ oder $k \neq l$ auf genau einer Geraden.

Es ergeben sich nun beim Viereckssatz mit eigentlichen Geraden die folgenden in Abb. 7 dargestellten verschiedenen Fälle:

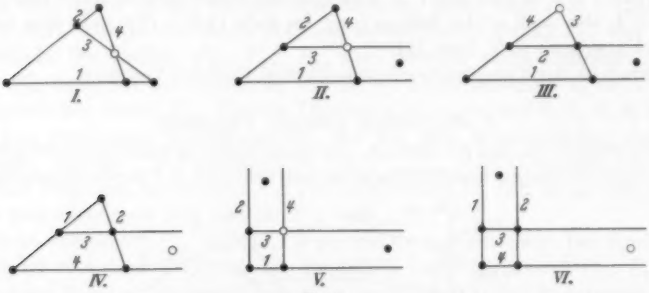


Fig. 7

I. Nach (b) schneiden sich die Lote $\bar{1}, \bar{2}, \bar{3}$ und $\bar{1}, \bar{2}, \bar{4}$ in je einem, also im gleichen Punkt. Nach (a) folgt somit die Behauptung.

II. Wieder schneiden sich nach (b) $\bar{1}, \bar{2}, \bar{4}$ in einem Punkt. Ferner ergibt (c), daß $\bar{1} = \bar{3}$ ist. Dann liefert erneut (a) die Behauptung.

III. Man verbinde gemäß (e) die Punkte $(1, 3)$ und $(2, 4)$ durch eine Gerade 5. Wir nehmen zuerst an, die zugehörige Gerade a_5 aus $\Pi(A)$ verläuft nicht durch B, d. h. $(13) \neq (24)$. Dann geht die entsprechende Gerade b_5 von $\Pi(B)$ auch nicht durch A. Wir können also (b) anwenden und erhalten, daß $\bar{1}, \bar{4}, \bar{5}$ durch einen Punkt verlaufen. Dieser fällt wegen $\bar{1} = \bar{2}$ [vgl. (c)] mit dem Schnittpunkt von $\bar{2}, \bar{3}, \bar{5}$ zusammen. (a) ergibt dann die Behauptung. Geht

aber a_5 durch B, so verbinden wir (1, 4) und (2, 3) durch eine Gerade 6 und wenden einen entsprechenden Schluß an. Geht auch der zugehörige Kreis a_6 durch B, so folgt (14) = (23), so daß unter den Kreisen (ik) nur vier verschiedene sind. Dann gibt es nach (18) einen Punkt P der Gerade a_4 von $\Pi(A)$, der auf keinem der vier Kreise (ik) liegt. Wir verbinden ihn mit B durch einen Kreis (oo). Der durch (oo) und b_4 bestimmte Punkt von $\Pi(B)$ sei Q. Im Schema ergibt (P, Q) einen neuen Punkt auf 4 (vgl. Abb. 8). Man verbindet diesen mit (13) und (23). In den entstehenden Dreiecken läßt sich (b) mehrfach anwenden und gestattet den Schluß, daß $\bar{2}, \bar{3}, \bar{4}$ sich in einem Punkt schneiden, was nach (a) wieder die Behauptung ergibt.

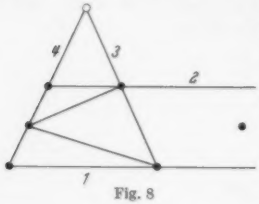


Fig. 8

IV. Nach (a) schneiden sich $\bar{1}, \bar{2}, \bar{3}$ in einem Punkt, ebenso $\bar{1}, \bar{2}, \bar{4}$. Diese Punkte sind einander gleich, so daß wegen (d) $\bar{3} = \bar{4}$ folgt. Die Behauptung ist dann nach (c) richtig.

V. Man verbinde wie in III. (1, 4) und (2, 3) durch eine Gerade 5. Ist (14) $\neq$ (23), dann treffen sich $\bar{1}, \bar{2}, \bar{5}$ nach (b) in einem Punkt. Wegen (c) gilt aber $\bar{1} = \bar{3}$ und $\bar{2} = \bar{4}$. Somit folgt die Behauptung aus (a). Im Falle (14) = (23) findet man auf a_4 wieder wie in III. einen Punkt, der keiner der Geraden (ik) angehört, und kommt auf einem entsprechenden Umweg zum Ziel.

VI. Wie in V. führe man die Hilfsgerade 5 ein. Ist (14) $\neq$ (23), dann sieht man, daß $\bar{1} = \bar{2}$ ist und sich $\bar{1}, \bar{3}, \bar{4}$ im gleichen Punkt treffen. Dann folgt nach (d) $\bar{3} = \bar{4}$, also nach (c) die Behauptung. Im Falle (14) = (23) geht man wieder einen Umweg wie in V. bzw. III.

Damit ist der Viereckssatz mit eigentlichen Geraden bewiesen.

2. Der einfach entartete Viereckssatz

Wir nehmen an, daß der Kreis a_1 „zu einem Nullkreis zusammenschrumpft", d. h., daß die entsprechende Gerade des betrachteten A-Vierecks die uneigentliche Gerade von $\Pi(A)$ wird. Die Punkte P_{12}, P_{13}, P_{14} sind dann uneigentlich und werden durch die Berührfährten repräsentiert, die bzw. von den Kreisen

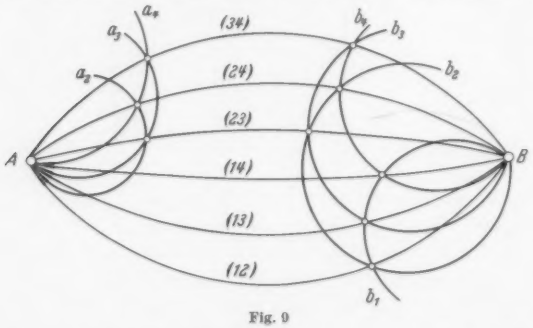


Fig. 9

a_2, a_3, a_4 in A bestimmt werden. Die Punkte P_{23}, P_{24}, P_{34} sind mithin eigentlich (Abb. 9).

Seien zuerst die Paare P_{1k}, Q_{1k} $A\,B$-perspektiv ($k = 2, 3, 4$). Das A-Lot $\bar{a}_i$ ($i = 2, 3$ oder 4) ist nach (8) zu ($1\,i$) orthogonal, also auch zu dem Kreis b_1, der in dem von b_i und ($1\,i$) aufgespannten Büschel liegt. Wegen (9) liegen daher die $\bar{a}_i$ in einem Büschel. Der Satz folgt somit aus dem ersten Dreiecklotsatz.

Wir nehmen jetzt an, die Paare P_{23}, Q_{23} bzw. P_{24}, Q_{24} bzw. P_{34}, Q_{34} sind je $A\,B$-perspektiv. Dann folgt mit Hilfe des zweiten Dreiecklotsatzes, daß die A-Lote $\bar{a}_2, \bar{a}_3, \bar{a}_4$ in einem Büschel liegen. Zwei von ihnen erkennt man wie eben als zu b_1 orthogonal; dies gilt nach (O 3) mithin für den dritten. Die Behauptung folgt jetzt aus (12).

3. Der zweifach entartete Viereckssatz

Wir haben zwei wesentlich verschiedene Fälle zu unterscheiden: a) Die Punkte P_{1i} und Q_{1i} sind uneigentlich ($i = 2, 3, 4$). b) Die Punkte P_{1i} ($i = 2, 3, 4$) und Q_{12}, Q_{23}, Q_{24} sind uneigentlich.

Fall a). Man schließt genau wie im Falle des einfach entarteten Viereckssatzes, wobei lediglich $\bar{a}_i \perp b_1$ durch $\bar{a}_i \ni B$ zu ersetzen ist ($i = 2, 3, 4$).

Fall b). Man hat verschiedenartige Schließungssätze, je nachdem, welches der Paare P_{ik}, Q_{ik} als $A\,B$-perspektiv nachzuweisen ist. Wir zeigen jedoch zuerst, daß alle diese Sätze äquivalent sind[8]).

Sei etwa P_{24}, Q_{24} als $A\,B$-perspektiv nachzuweisen. Angenommen, der Kreis (24) durch A, Q_{24} berührt den Kreis a_2, dann folgt (12) = (24) entgegen (6.1). Also ist der durch (24) und a_2 bestimmte Punkt P'_{24} von $\Pi(A)$ eigentlich. Die Punkte P_{23}, P'_{24}, P_{34} sind offenbar nicht kollinear. Durch P_{34}, P'_{24} legen wir die Gerade a'_4 von $\Pi(A)$. Dann bestimmt a'_2 in A einen uneigentlichen Punkt P'_{14}. Für die Vierecke a_1, a_2, a_3, a'_4 und b_1, b_2, b_3, b_4 haben wir nun denjenigen Schließungssatz, der aus dem ursprünglichen dadurch entsteht, daß die $A\,B$-Perspektivität von P_{14}, Q_{14} statt derjenigen von P_{24}, Q_{24} eine Folge aus den übrigen $A\,B$-Perspektivitäten ist. Gilt dieser Satz, dann folgt rückwärts nach (4a) und (6) $P'_{14} = P_{14}$; $P'_{24} = P_{24}$.

Seien umgekehrt P_{14}, Q_{14} als $A\,B$-perspektiv nachzuweisen. Der Kreis (14) bestimmt in A einen uneigentlichen Punkt P'_{14} von $\Pi(A)$. Durch P'_{14}, P_{34} lege man einen Kreis a'_4. Berührten sich a'_4, a_2, so folgte (14) = (12) entgegen (6.1). Nun wendet man entsprechende Schlüsse wie eben an.

Bei diesen Überlegungen haben wir nicht benutzt, welche der Punkte Q_{ik} uneigentlich sind. Durch mehrfache Anwendung der gleichen Schlüsse auf die anderen Paare eigentlicher Punkt — uneigentlicher Punkt des A-Vierecks folgt also die behauptete Äquivalenz der betrachteten Sätze.

Da in dem A-Viereck und dem B-Viereck des Satzes je eine Seite als uneigentlich angenommen ist, sind die Diagonalen sämtlich eigentlich. Verlaufen im A-Viereck die Diagonalen $P_{12}P_{34}$ und $P_{13}P_{24}$ nicht durch B [d. h. ist

[8]) Einen entsprechenden Sachverhalt hat man bei dem Viereckssatz mit eigentlichen Geraden nicht.

$(12) \neq (34)$ und $(13) \neq (24)$], so sei P ihr Schnittpunkt. Dann gehen die entsprechenden Diagonalen des B-Vierecks nicht durch A; ihr Schnittpunkt sei Q. Enthält aber eine Diagonale des A-Vierecks den Punkt B, so sind höchstens 5 der Kreise (ik) voneinander verschieden. Es gibt also nach (18) etwa auf a_4 einen Punkt P', der keinem der Kreise (ik) angehört. Wir verbinden P' mit B durch einen Kreis (oo). Q' sei der durch (oo) und b_4 bestimmte Punkt von $\Pi(B)$.

Nach dem eben Bewiesenen können wir annehmen, daß die AB-Perspektivität von P_{14}, Q_{14} nachzuweisen ist. In Abb. 10 zeichnen wir jeweils

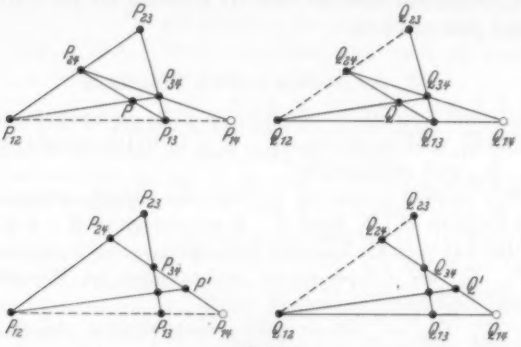


Fig. 10

schematisch das A- und das B-Viereck nebeneinander, wobei die uneigentlichen Geraden durch punktierte Linien ausgedrückt werden. Man kann daraus ablesen, wie aus dem einfach entarteten Viereckssatz zuerst die AB-Perspektivität von P, Q bzw. P', Q' und dann diejenige von P_{14}, Q_{14} folgt.

7. Die Möbiusebenen $\mathfrak{M}(\mathfrak{R}, \mathfrak{S})$

Im Beweis des Viereckssatzes ist bereits angedeutet, wie wir mit Hilfe der Kreise eine räumliche projektive Geometrie aufbauen:

Seien A, B zwei festgewählte Punkte mit $A \neq B$. Wir betrachten die projektiven Unterebenen $\Pi(A)$ und $\Pi(B)$ der Möbiusebene. Jedem AB-perspektiven Paar von Punkten P, Q mit $P \in \Pi(A)$ und $Q \in \Pi(B)$ ordnen wir einen *Punkt* (P, Q) einer räumlichen Geometrie zu. Den Geraden von $\Pi(A)$ und $\Pi(B)$ ordnen wir *Ebenen* α, β, ... zu. Ist $A \in \alpha$ und $B \notin \alpha$, so nennen wir α eine *A-Ebene*; ist $A \notin \beta$ und $B \in \beta$, so heiße β eine *B-Ebene*. Ist A, $B \in \alpha$, so sprechen wir von einer *AB-Ebene*. Wir sagen: (P, Q) *liegt auf* α, wenn $P \in \alpha$ oder $Q \in \alpha$ gilt. Die Schnittgebilde verschiedener Ebenen heißen *Geraden* (eine Verwechslung mit den Geraden von $\Pi(A)$ bzw. $\Pi(B)$ ist nicht zu befürchten). Den Punkten A, B ordnen wir je einen Punkt der räumlichen Geometrie zu. A liege auf allen A- und AB-Ebenen, B auf allen B- und AB-Ebenen. Wir nennen A, B auch *Grundpunkte*. Für die so erklärten Punkte, Grundpunkte und Ebenen weisen wir folgende Eigenschaften $\langle 1\rangle$—$\langle 5\rangle$ nach:

Nach Definition hat man:

⟨1⟩ *Auf jeder Ebene liegt ein Grundpunkt.*

Aus den Eigenschaften von $\Pi(A)$ bzw. $\Pi(B)$ folgt:

⟨2⟩ *Durch einen Grundpunkt und zwei von ihm verschiedene Punkte geht stets eine Ebene.*

⟨3⟩ *Drei Ebenen enthalten stets einen gemeinsamen Punkt.*

Beweis. ξ, η, ζ seien die Ebenen. Zwei von ihnen verlaufen durch den gleichen Grundpunkt, etwa ξ, η durch A. Geht auch ζ durch A, so ist A ein gemeinsamer Punkt der Ebenen. Sei daher $B \in \zeta$; $A \notin \zeta$. Falls nicht schon $\xi = \eta$ ist, bestimmen ξ, η eine Fährte oder Berührfährte (= Punkt von $\Pi(A)$). Man verbinde diese mit B durch einen Kreis a. Dann bestimmen a und ζ eine Fährte oder Berührfährte, die mit der eben genannten Fährte bzw. Berührfährte ein $A\,B$-perspektives Paar von Punkten aus $\Pi(A)$ bzw. $\Pi(B)$ darstellt. Diesem ist ein Schnittpunkt von ξ, η, ζ zugeordnet, und zwar genau einer.

⟨4⟩ *Enthalten drei verschiedene Ebenen zwei verschiedene gemeinsame Punkte, so besitzen sie zu je zweien den gleichen Durchschnitt.*

Beweis. Seien (P, Q), (P', Q') die angenommenen Punkte; $P, P' \in \Pi(A)$; $Q, Q' \in \Pi(B)$. Gehen die Ebenen nicht alle durch den gleichen Grundpunkt, so haben sie, wie wir im Beweis von ⟨3⟩ sahen, genau einen Schnittpunkt im Widerspruch zur Annahme. Sei also etwa A in allen drei Ebenen ξ, η, ζ gelegen. Wäre $P \neq P'$, so bestimmte die Gerade $P\,P'$ von $\Pi(A)$ genau eine A-Ebene; es folgte also $\xi = \eta = \zeta$. Folglich gilt $P = P'$. Ist Q'' irgendein zu $P\,A\,B$-perspektiver Punkt aus $\Pi(B)$, dann liegt (P, Q'') in allen drei Ebenen. Auf diese Weise erhält man sämtliche gemeinsamen Punkte der drei Ebenen.

⟨5⟩ *Durch jeden Punkt verlaufen drei Ebenen, die nur diesen Punkt gemein haben. Es gibt zwei Grundpunkte und drei weitere Punkte, so daß keine vier dieser fünf Punkte komplanar sind.*

Beweis. Der erste Teil von ⟨5⟩ ergibt sich unmittelbar aus der Definition der Punkte und — im Falle der Grundpunkte — aus der Tatsache, daß durch A bzw. B je drei Kreise a, b, c mit $a \cap b \cap c = \{A\}$ bzw. $= \{B\}$ verlaufen. Mit Hilfe dieser Kreise sieht man auch den zweiten Teil von ⟨5⟩ ein.

Die Punkte und Ebenen bilden somit ein „*2-Gefüge*" (siehe die Arbeit [5] des Verf.). In diesem 2-Gefüge läßt sich der Viereckssatz folgendermaßen deuten: Sind $\varepsilon, \lambda, \mu, \nu$ vier Ebenen, dann verstehen wir unter dem Symbol „$\varepsilon\,\lambda\,\mu\,\nu$" die Tatsache: „$\varepsilon, \lambda, \mu, \nu$ schneiden sich in einem und nur einem Punkt". Dann gilt:

Satz 3. *(Oktaedersatz) Sind $\alpha, \beta, \gamma, \delta$ vier verschiedene A-Ebenen, $\alpha', \beta', \gamma', \delta'$ vier verschiedene B-Ebenen, dann folgt aus den Beziehungen $\alpha\,\beta\,\gamma\,\delta$, $\alpha'\,\beta'\,\gamma'\,\delta'$, $\alpha\,\alpha'\,\delta\,\delta'$, $\alpha\,\alpha'\,\beta\,\beta'$, $\alpha\,\alpha'\,\gamma\,\gamma'$, $\beta\,\beta'\,\delta\,\delta'$, $\gamma\,\gamma'\,\delta\,\delta'$, daß auch $\beta\,\beta'\,\gamma\,\gamma'$ gilt* (Abb. 11).

Ein Satz der Arbeit [5] des Verf. besagt nun:

Satz 4. *Die Punkte und Geraden eines 2-Gefüges (Grundpunkte A, B) bilden, sofern der Oktaedersatz gilt, nach Einführung neuer mit A, B kollinearer Punkte und nach Definition geeigneter Geraden in den $A\,B$-Ebenen einen dreidimensionalen projektiven Raum $\mathfrak{R}$.*

Jeder Punkt $\neq A, B$ der Möbiusebene bestimmt mit A, B je genau eine Fährte, also einen Punkt $P \in \Pi(A)$ bzw. $Q \in \Pi(B)$. Auf diese Weise läßt sich jedem Punkt $\neq A, B$ der Möbiusebene genau ein Punkt (P, Q) von $\mathfrak{R}$ zuordnen. A, B hatten wir schon mit Punkten von $\mathfrak{R}$ identifiziert. Wir bezeichnen die so gewonnene Punktmenge von $\mathfrak{R}$ mit $\mathfrak{S}$. Die Kreise der Möbiusebene durch A oder B gehen in ebene Schnitte von $\mathfrak{S}$ über. Der Büschelsatz garantiert darüber hinaus, daß sämtliche Kreise der Möbiusebene auf ebene Schnitte von $\mathfrak{S}$ abgebildet werden (vgl. Abb. 12, die zum Ausdruck bringt, daß sich zwei Geraden stets dann schneiden, wenn sie beide das Bild eines Kreises p in verschiedenen Punkten treffen).

Wir definieren jetzt folgenden Begriff, der eine Verallgemeinerung des Begriffes

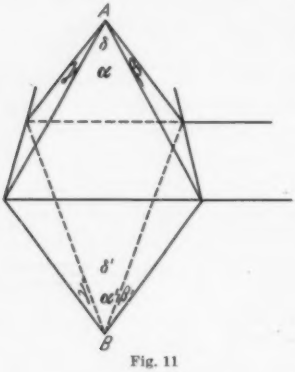


Fig. 11

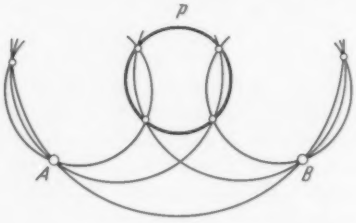


Fig. 12

der streng konvexen Fläche mit überall eindeutiger Stützebene darstellt: Sei eine Untermenge $\mathfrak{S}$ der Punkte eines projektiven Raumes $\mathfrak{R}$ gegeben, die folgende Eigenschaften besitzt:

a) Zu jedem $P \in \mathfrak{S}$ existiert genau eine Ebene ε_P mit $\varepsilon_P \cap \mathfrak{S} = \{P\}$.

b) Ist $P \in \mathfrak{S}$ und liegt P auf der Geraden g, dann liegt g entweder in ε_P oder trifft $\mathfrak{S}$ in einem Punkt $\neq P$. Dann nennen wir $\mathfrak{S}$ eine *schwach konvexe Semifläche*.

(7.1) *Die aus mehr als einem Punkt bestehenden ebenen Schnitte einer schwach konvexen Semifläche $\mathfrak{S}$ bilden eine Möbiusebene.*

Beweis. Wir erklären die aus mehr als einem Punkt bestehenden ebenen Schnitte von $\mathfrak{S}$ als *Kreise*, die Punkte von $\mathfrak{S}$ als *$\mathfrak{M}$-Punkte*. Da durch drei $\mathfrak{M}$-Punkte stets eine Ebene verläuft, gilt der erste Teil von (M I). Der zweite Teil folgt sofort aus der Definition der Kreise. (M II) sieht man so ein: Seien A, B $\mathfrak{M}$-Punkte, sei a ein Kreis, so daß $A \in a$, $B \notin a$ gilt. Nach a) hat man $B \notin \varepsilon_A$. Sei g die Schnittgerade von ε_A und der durch a bestimmten Ebene. Dann ist $B \notin g$ und bestimmen B, g genau eine Ebene, die einen Kreis b aus $\mathfrak{S}$ ausschneidet. Gemeinsame Punkte von a, b liegen auf g, also folgt wegen $g \cap \mathfrak{S} = \{A\}$ auch $a \cap b = \{A\}$. Wegen b) ist b eindeutig bestimmt. Dies beweist (M II). (M III) folgt aus der Definition der Kreise und b). — Damit ist (7.1) bewiesen.

Wir bezeichnen die Möbiusebene von (7.1) mit $\mathfrak{M}(\mathfrak{K}, \mathfrak{S})$, wo $\mathfrak{K}$ der zu $\mathfrak{R}$ gehörige Koordinatenschiefkörper ist. (Man vergleiche zu (7.1) die Arbeit [4] des Verf.)

Satz 5. *Die Möbiusebenen, die dem Viereckssatz und dem Büschelsatz genügen, sind genau die Ebenen* $\mathfrak{M}(\mathfrak{K}, \mathfrak{S})$.

Beweis. Daß die Ebenen $\mathfrak{M}(\mathfrak{K}, \mathfrak{S})$ dem Büschelsatz und dem Viereckssatz genügen, folgt einfach daraus, daß diese beiden Sätze zugleich Schließungssätze für Ebenen eines projektiven Raumes darstellen, wenn man nur „Kreis" durch „Ebene" ersetzt.

Umgekehrt haben wir beim Nachweis, daß die oben betrachtete Möbiusebene als System der ebenen Schnitte einer Menge $\mathfrak{S}'$ von Punkten von $\mathfrak{R}$ aufgefaßt werden kann, neben den Eigenschaften (M I)—(M III) nur den Viereckssatz und den Büschelsatz benutzt. Wir haben also nur noch zu zeigen, daß $\mathfrak{S}'$ die Eigenschaften a), b) besitzt: Sei $P \in \mathfrak{S}'$ beliebig. Jede Ebene ε durch P erfüllt entweder $\varepsilon \cap \mathfrak{S}' = \{P\}$ oder schneidet $\mathfrak{S}'$ in einem Kreis a. Im letzteren Fall enthält ε genau eine Gerade g mit $g \cap \mathfrak{S}' = \{P\}$; P und a bestimmen nämlich genau eine Berührfährte. Seien g, g' mit $g \neq g'$ zwei Geraden dieser Art. Angenommen, die durch g, g' bestimmte Ebene γ schneidet $\mathfrak{S}'$ in einem Kreis b. Dann lege man durch ein $Q \in \mathfrak{S}'$ mit $Q \notin b$ und g bzw. g' je eine Ebene. Diese Ebenen schneiden $\mathfrak{S}'$ in verschiedenen Berührkreisen von b entgegen (M II). γ ist also gerade die gesuchte Berührebene ε_P und es gelten a), b). — Damit ist Satz 5 bewiesen.

8. Die Möbiusebenen $\mathfrak{M}(\mathfrak{K}, \mathfrak{Q})$

Zur Kennzeichnung der Ebenen $\mathfrak{M}(\mathfrak{K}, \mathfrak{S})$ hatten wir neben den Eigenschaften (M I)—(M III) der Möbiusebenen nur den Büschelsatz und den Viereckssatz, also ausschließlich Inzidenzeigenschaften, herangezogen. Wir legen jetzt wieder eine (MO)-Ebene zugrunde, die dem Schließungssatz (S) genügt. Da wir den Büschelsatz und den Viereckssatz mit Hilfe von (S) hergeleitet hatten, bildet diese Ebene eine Ebene $\mathfrak{M}(\mathfrak{K}, \mathfrak{S})$ (wobei der im 4. Abschnitt behandelte Fall gesondert zu betrachten ist). Wir zeigen, daß $\mathfrak{S} = \mathfrak{Q}$ eine Quadrik (siehe Einleitung) ist.

Ist a ein Kreis von $\mathfrak{M}(\mathfrak{K}, \mathfrak{S})$, dann bezeichnen wir mit ε_a die Ebene, die a aus $\mathfrak{S}$ ausschneidet. Wir betrachten nun alle Kreise $b, b', \ldots$ mit $b, b', \ldots \perp a$. Mit Hilfe von (12) sieht man, daß sich die Schnittgeraden $\varepsilon_b \cap \varepsilon_{b'}, \ldots$ alle in einem Punkt P des projektiven Raumes $\mathfrak{R}$ (über $\mathfrak{K}$) treffen. Wir nennen P den *Pol* von ε_a und bezeichnen ε_a als *Polarebene* von P.

Sei R jetzt irgendein Punkt von $\mathfrak{R}$. Wir finden gewiß drei Kreise b, b', b'' [etwa mit Hilfe von (M III)], die sich zu je zweien schneiden und $\varepsilon_b \cap \varepsilon_{b'} \cap \varepsilon_{b''} = \{R\}$ erfüllen. Q, Q', Q'' seien bzw. die Pole von $\varepsilon_b, \varepsilon_{b'}, \varepsilon_{b''}$. Gehörten Q, Q', Q'' einer Geraden an, so gäbe es verschiedene gemeinsame Orthogonalkreise von b, b', b''. Dies führte nach (11) dazu, daß b, b', b'' in einem Büschel lägen entgegen $\varepsilon_b \cap \varepsilon_{b'} \cap \varepsilon_{b''} = \{R\}$. Somit bestimmen Q, Q', Q'' eine Ebene; wir nennen sie *Polarebene* von R. Besitzen b, b', b'' einen gemein-

samen Orthogonalkreis, so stimmt diese Definition mit der oben gegebenen überein.

Wir zeigen jetzt, daß die Polarebene von R eindeutig bestimmt ist. Sei c, c', c'' ein zweites Tripel von Kreisen mit $\varepsilon_c \cap \varepsilon_{c'} \cap \varepsilon_{c''} = \{R\}$, so daß sich c, c', c'' zu je zweien schneiden. T, T', T'' seien bzw. die Pole von $\varepsilon_c, \varepsilon_{c'}, \varepsilon_{c''}$. Etwa $\varepsilon_b \cap \varepsilon_{b'}$ und $\varepsilon_c \cap \varepsilon_{c'}$ bestimmen dann eine Ebene, die aus $\mathfrak{S}$ einen Kreis d oder einen Punkt D ausschneidet. Der Pol von ε_d bzw. D selbst stellt dann einen Schnittpunkt der Geraden QQ' und TT'' dar. Auf diese Weise sieht man, daß sich je zwei der Geraden QQ', QQ'', $Q'Q''$, TT', TT''', $T'T'''$ schneiden. Also bestimmen Q, Q', Q'' und T, T', T''' die gleiche Ebene.

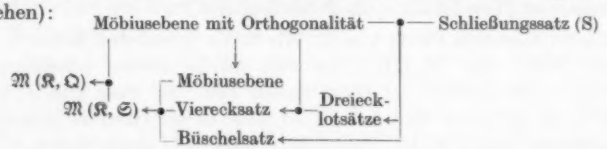
Sei $\hat{b}$ irgendein Kreis, so daß der Pol von $\varepsilon_{\hat{b}}$ in der Polarebene von R liegt. Wir weisen nach, daß $\varepsilon_{\hat{b}}$ durch R verläuft. Sei B ein beliebiger Punkt von $\hat{b}$. Wir legen durch B Kreise b_1 bzw. b_2 bzw. b_3, die mit b, b' bzw. b, b'' bzw. b', b'' in einem Büschel liegen. Zwei dieser Kreise sind gewiß verschieden, etwa b_1, b_2. Liegen dann $b_1, b_2, \hat{b}$ nicht in einem Büschel, so bestimmen die Pole von ε_{b_1}, ε_{b_2}, $\varepsilon_{\hat{b}}$ eine Ebene, und zwar die Polarebene von B. Es folgt also $B = R$. Da aber B beliebig $\in b$ gewählt war, führt dies zu einem Widerspruch. Also liegen $b_1, b_2, \hat{b}$ in einem Büschel, d. h. $\varepsilon_{\hat{b}}$ geht durch R.

Ist g eine beliebige Gerade der Polarebene von R, so lege man durch g zwei Ebenen ε_{b_1}, ε_{b_2}, die $\mathfrak{S}$ in Kreisen b_1, b_2 schneiden. Sind a_1, a_2 Kreise mit $a_1, a_2 \perp b_1, b_2$, so liegen die Pole von ε_{a_1}, ε_{a_2} auf g, also verlaufen ε_{a_1}, ε_{a_2} nach dem eben Bewiesenen durch R. Auf diese Weise sieht man: Liegt S auf der Polarebene von R, dann liegt R auf der Polarebene von S. (Man lege durch S zwei Geraden der Polarebene von R.)

Nach H. LENZ [6] ist die so gewonnene Polarität eine Polarität von $\mathfrak{R}$ im Sinne der analytischen Geometrie (siehe auch R. BAER [1]). Die Punkte von $\mathfrak{S}$ sind genau die Punkte von $\mathfrak{R}$, die mit ihrer Polarebene inzidieren, d. h. $\mathfrak{S}$ ist eine Quadrik $\mathfrak{Q}$. Aus den Eigenschaften (M I)—(M III) einer Möbiusebene und a), b) (Abschnitt 7) folgt, daß $\mathfrak{Q}$ schwach konvex ist (siehe Einleitung).

Im 4. Abschnitt war in unsere Voraussetzungen die Bedingung (Z) eingegangen. Falls (Z) nicht gilt, hatten wir schon die Richtigkeit des Hauptsatzes (siehe Abschnitt 1) erläutert. Falls (Z) gilt, folgt jetzt ebenfalls der Hauptsatz.

Das folgende Schema gibt einen Überblick über die Abhängigkeit verschiedener in dieser Arbeit vorkommender Sätze. Dabei ist jeweils angegeben, was beim Beweis eines Satzes benutzt wurde. (Die verschiedenen Möbiusebenen des Schemas sind als Gesamtheiten ihrer definierenden Eigenschaften zu verstehen):



Es ergeben sich daraus neben der im Hauptsatz genannten noch weitere Kennzeichnungen der Möbiusebenen $\mathfrak{M}(\mathfrak{R}, \mathfrak{Q})$:

Satz 6. *Eine (MO)-Ebene $\mathfrak{M}$ ist genau dann eine Ebene $\mathfrak{M}(\mathfrak{K}, \Omega)$, wenn sie eine der folgenden Bedingungen erfüllt:*

(a) *$\mathfrak{M}$ genügt dem Schließungssatz (S).*

(b) *$\mathfrak{M}$ genügt den Dreiecklotsätzen und dem Büschelsatz.*

(c) *$\mathfrak{M}$ genügt dem Viereckssatz und dem Büschelsatz.*

(d) *$\mathfrak{M}$ ist eine Ebene $\mathfrak{M}(\mathfrak{K}, \mathfrak{S})$.*

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On the spectral theory of elliptic differential operators. I*

By

FELIX E. BROWDER in New Haven, Connecticut

Introduction

Let A be a linear elliptic partial differential operator of order $2m$ defined on a (possibly unbounded) domain G of Euclidean n-space. If we are given a space of functions on G, by a realization of A in that function-space under a given set of null boundary conditions we shall mean the restriction of A to the sub-class of functions which satisfy that set of boundary conditions in some suitable sense.

The study of the spectral properties of the realizations of A in various function spaces on G (i.e. the spectra, resolvents, adjoints, and spectral decompositions of such operators) is a natural point of confluence of at least three distinct types of problems of importance in the theory of partial differential equations, the solvability of boundary value problems for the elliptic operator A, the existence of eigenfunction expansions and other types of spectral decompositions for realizations of A, and the existence, uniqueness, and differentiability of solutions of initial-boundary value problems for equations of the form $\frac{\partial u}{\partial t} = -A(t)u$ and $\frac{\partial^2 u}{\partial t^2} = -A(t)u$. In the first instance, the connection between spectral theory and the solvability of boundary value problems is founded upon the observation, usually attributed to Fredholm, that the appropriate question to ask may not be, whether $Au = f$ has a solution under our given set of boundary conditions, but rather, for what values of the complex parameter ζ does $(Au - \zeta u) = f$ have one and only one solution for each f, i.e. which ζ lie in the resolvent set of A for which the resolvent $(A - \zeta I)^{-1}$ exists. Under more special conditions, if the range of A is closed in the space considered (in particular, if for some ζ the resolvent $(A - \zeta I)^{-1}$ is compact), the set of f for which solutions of $Au = f$ exist is the orthogonal complement of the solutions v in the dual space of the equation $A^*v = 0$, and the problem of determining the form of A^* makes its appearance. In the second instance, that of spectral decompositions and eigenfunction expansions, one has first of all in the case of symmetric problems the task of determining whether their realizations in a given Hilbert space are Hermitian so that the classical spectral theory of

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Hermitian operators can be applied, i.e. one must show that $A^* = A$. In non-symmetric problems, one may hope by a study of the adjoint A^* of A and the resolvent $(A - \zeta I)^{-1}$ to show that the realization operator A falls within some general class of operators for which generalized spectral decompositions have been constructed, and most conspicuously the spectral operators of DUNFORD. In the third instance, the solution of initial boundary value problems, the theory of generation of semi-groups of linear operators due to HILLE, YOSIDA, FELLER, PHILLIPS, KATO, and others provides a general mechanism by which one may pass directly from information about the resolvent of A to the solution of initial-value problems.

In a recent series of papers, ([9], [10], [11]), the writer has studied the problem of constructing eigenfunction expansions in terms of eigenfunctions of non-symmetric linear partial differential operators. We have constructed a theory of such expansions for several classes of realizations of differential operators, for subnormal realizations in [9] and for decomposable realizations in [10], but the problem has remained of showing that the simplest realizations of some general family of differential operators fall within one of these classes and, more generally, of studying in detail the spectral structure of such simple realizations.

It is the purpose of the present paper to begin this study with the analysis of what is, in principle, the simplest general class of realizations of partial differential operators, the elliptic operators (not necessarily self-adjoint) under null Dirichlet boundary conditions on an unbounded open subset G of Euclidean n-space. In the following paper of this series, we shall extend our results to the more general class of regular boundary value problems for elliptic operators, study the spectral theory in the spaces of Hölder-differentiable functions, obtain sharper (though more complicated) estimates in the L^p-spaces, and apply the ensemble of our results to the study of initial-value problems. Most of the results which are established here have been announced by the writer in [7], [8], [14], and [15].

An important aspect of our results which deserves emphasis at this point is that our study is carried through under minimal assumptions on the smoothness or boundedness of the coefficients of A on G. In particular, the existence theorems for solutions of boundary value problems which are derived hold under extremely general conditions, and in the case of bounded domains under conditions in which the standard variational approach cannot be applied. Though we do not emphasize this particular feature in the context of the present discussion, important applications may be obtained of such results to the theory of boundary value problems for non-linear equations, both elliptic and parabolic. The basic technical apparatus which underlies our theory, the construction of a-priori estimates for solutions of elliptic equations in the spaces $W^{2m,p}(G)$ and $C^{2m,\lambda}(G)$, has an even more crucial role to play in the non-linear theory. We discuss the character and the history of such a-priori estimates in more detail below, in connection with our summary of the discussion of Section 2.

Section 1 is concerned with the definition and properties of the function spaces $W^{2m,p}(G)$ and $C^{2m,\lambda}(G)$ which are the basic sub-stratum on which our later arguments are founded. The definition of the spaces $W^{j,p}(G)$ and the discovery of their basic properties are due to S. L. SOBOLEV ([65], [66]). Most of the preceding discussions of such spaces have been devoted to the case of bounded G. Our principal concern in Section 1 is to establish the basic properties of these spaces without boundedness assumptions. Lemma 5 contains the result which is usually referred to as Sobolev's Imbedding Theorem. For general p and bounded G, Lemma 6 is due to E. GAGLIARDO [27], though the special case $p = 2$ was already treated by the writer in [12].

In Section 2, we come to the definition and properties of the Dirichlet problem. We assume that the linear elliptic differential operator A of order $2m$ with which we are concerned satisfies two regularity and boundedness conditions on its coefficients:

(I) *All the coefficients are essentially bounded on G;*

(II) *The top-order coefficients are uniformly continuous on G.*

A itself is assumed to be uniformly elliptic as well as regularly elliptic in the sense of Definition 5. (In the case when the top order coefficients are real-valued functions, this second condition is vacuous.) Definitions 3 and 4 make precise the concept of the realization A_p of A in $L^p(G)$ under null Dirichlet boundary conditions of order m. Theorem 1 asserts that such realizations A_p are always closed operators in $L^p(G)$ under our given hypotheses on A, and is derived from the stronger quantitative result of Theorem 2, which is the basic a-priori bound on elliptic boundary-value problems in $L^p(G)$. In brief, Theorem 2 asserts that if $D^\alpha u$ is any derivative of u of order $\leqq 2m$, and if u is a function in $W^{2m,p}(G)$ which satisfies the null Dirichlet boundary conditions of order m on the boundary of G, then there exists a constant K, independent of u, such that $\|D^\alpha u\|_{L^p(G)} \leqq K\{\|A_p u\|_{L^p} + \|u\|_{L^p}\}$. Estimates of the type of Theorem 2 were first obtained for the case $p = 2$, $m = 1$, by SCHAUDER [57], and somewhat later by CACCIOPOLI [18] and LADYZENSKAYA [43]. For general m and $p = 2$, such estimates were given for strongly elliptic operators by GUSEVA [33] and BROWDER [6], and in a slightly weaker form by NIRENBERG [52]. A corresponding result with $p = 2$ for second-order strongly elliptic systems had been obtained earlier by C. B. MORREY [49]. For general p, $m = 1$, and the Dirichlet problem for a strongly elliptic operator, an estimate of the type of Theorem 2 was given by KOSHELEV [40]. L^2-estimates in the Dirichlet problem for operators A which are elliptic but not strongly elliptic were given by M. SCHECHTER [58]. Theorem 2 is a special case of a general class of a-priori estimates of the same type for a general family of boundary value problems for regularly elliptic differential operators, a family which the writer has called the regular boundary problems in [14]. Results on estimates in L^p and in Hölder norm have been announced for the regular problems in BROWDER [14], (cf. also Bull. Am. Math. Soc. 1956, p. 382), and by AGMON,

DOUGLIS, and NIRENBERG (jointly)[1]), and L^2-estimates have been given by M. SCHECHTER [61]. The corresponding result for the Dirichlet problem itself has been given in $L^p(G)$ by KOSHELEV [41]. The analytical details of the estimation (involving the explicit construction of the corresponding solutions of equations with constant coefficients on half-spaces as outlined in Section 2 below) are given in the writer's papers [16] and [17].

One of the essential differences between the argument given below for Theorem 2 and most of the discussions mentioned above is that we do not restrict our attention to bounded domains, but consider general domains G with a uniformly smooth boundary.

We continue in Section 2 with the consideration of a wider class of realizations of an elliptic operator A, those which satisfy the null Dirichlet boundary conditions of order m on a portion Γ_0 of the boundary Γ of G, while on the complementary part of Γ, $\Gamma_1 = \Gamma - \Gamma_0$, we have either the null Cauchy boundary conditions or, in the opposite case, no boundary conditions at all. The study of these realizations of A, in addition to its own intrinsic interest, is essential to obtain the finer results on the adjointness relations of Section 3. The remainder of Section 2 is devoted to the development of further tools for the argument of Section 3.

Section 3 is devoted to studying the adjoint of the realization A_p of an elliptic operator A in $L^p(G)$ under null Dirichlet boundary conditions. Its principal result under suitable hypotheses is that $(A_p)^* = A'_q$, where A' is the formal adjoint of A, A'_q is the realization of A' in $L^q(G)$ under null Dirichlet boundary conditions, and q is the conjugate exponent of p, $q = (p-1)^{-1}p$. The regularity condition involved in the hypotheses of Section 3 is the following:

(IV) *There exists an open set G_0 of E^n containing G with $d(G, E^n - G_0) > 0$ such that A is defined and regularly elliptic on G_0, the coefficients of A and of its adjoint A' are essentially bounded on G_0, and the top-order coefficients of A are uniformly continuous on G_0.*

If $p = 2$, the regularity condition (IV) is enough by itself to imply that $A_2^* = A'_2$, which is established in a more general form in Theorem 5. For the conclusion for general p, a somewhat more involved argument is necessary with an additional hypothesis of several alternate types. As a preliminary step, we establish in Theorem 9 the general conclusion under the hypotheses that the coefficients of A and A' satisfy local Lipschitz conditions, while A and A' have local uniqueness of the Cauchy problem in the small. (It follows from recent examples due to P. COHEN and to PLIS that this latter assumption does not hold in general for elliptic operators with coincident complex characteristic roots.) Using the techniques of Theorem 10, we then show in Theorem 12 that our adjointness theorem is valid without Lipschitz conditions or uniqueness of the Cauchy problem in the small provided that on every bounded subdomain of G, the realization of the top-order homogeneous part A^0 of A in L^2 under

[1]) At the moment when this paper was being submitted for publication, the paper of AGMON, DOUGLIS, and NIRENBERG has appeared in Comm. Pure App. Math. vol. 12, No. 4, Nov. 1959, pp. 623—730.

null Dirichlet conditions has at least one point in its resolvent set. This latter hypothesis is established in Section 4 for all strongly elliptic operators, and thereby the adjointness theorem for such operators. Finally in Theorem 14, the adjointness theorem is proved once more with the assumption of local Lipschitz conditions on the coefficients but without the unpleasant assumption of uniqueness of the Cauchy problem in the small.

In Section 4, we pass from the general class of elliptic operators with complex coefficients to the subclass of strongly elliptic equations, which includes in particular all elliptic operators with real coefficients. Theorem 15, which considers the case of an operator A which merely satisfies the regularity conditions (I) and (II) on its coefficients, obtains bounds from below for $\|(A_2 + \xi I)u\|$, which in the case of a bounded domain show that any ray making an acute angle with the negative real axis lies eventually in the resolvent set of A for any p. Under the stronger regularity assumption (IV), Theorem 16 obtains the adjointness theorem, together with the fact that A_2 is semi-bounded from below and has its spectrum contained in a half-plane. On the real axis, A_2 satisfies the Hille-Yosida bound on its resolvent. Theorem 17 specializing the conclusions of Theorem 15 to a bounded set G shows that the resolvent of A_p is a compact operator with the consequent discreteness of spectrum, finite-dimensionality of the eigenspaces, and the Fredholm alternative. In Theorem 18, it is shown that the resolvent in this case is an integral operator with a suitable iterate of finite trace class. In Theorem 19, it is assumed that A is essentially real, i.e. that its top-order coefficients are real-valued functions, with correspondingly sharper bounds on the spectrum and the resolvent. In particular, in Theorem 20, the spectrum lies within an algebraic curve with slope going to zero containing the positive real axis. The bounds obtained in this theorem for the resolvent of A_2 are sufficient to show the differentiability of the semi-group of which the operator A_2 is the infinitesimal generator. In the following paper, we shall give a partial extension to the operator A_p, which is sufficient to obtain differentiable semi-groups with A_p as generator, and give a detailed analysis of the results so obtained.

In Sections 1 to 4, we have assumed the bounds (I) and (II) on the coefficients of A. In Section 5, we modify these assumptions by considering a pertubation of A by lower order terms with singular or unbounded coefficients. Theorems 21, 23 and 24 give an extension of the adjointness theorem to such perturbed operators and thereby generalize results of KATO [38], STUMMEL [67], and WIENHOLTZ [70]. These results are generalized in three respects, to $m \geqq 1$ from $m = 1$, from $p = 2$ to general p, and finally with respect to the minimal assumption of smoothness of the coefficients. Because of the latter, it is necessary to carry through a detailed argument establishing a number of analytical inequalities involving terms of the form (Bf, g), where B satisfies the condition (IV), or the weaker regularity condition (III). These inequalities would follow trivially if we knew the existence of a bounded Dirichlet form, but are proved in the absence of such a form. In Section 7, we observe that a number of results for second-order operators follow simply and in a sharp form from the

fact that (IV) implies the existence of a bounded Dirichlet form for second-order operators. In particular, we obtain an extension of Wienholtz's theorem for second-order operators stating that with locally smooth coefficients, A_2 is essentially self-adjoint if it is semi-bounded. (Wienholtz's original result concerned the case in which $G = E^n$.) We might remark that the writer obtained the general adjointness theorem in $L^2(G)$ independently of Wienholtz's paper, announcing the result and some of its elaborations in the two Comptes Rendus notes [7] and [8].

Returning to Section 5, Theorem 26 considers the operator $A_2 + V(x)I$, with $V(x) \to \infty$ as $x \to \infty$ in G and shows that this operator has a compact resolvent. In Theorem 27, with a further assumption on $V(x)$ at infinity, it is shown that the resolvent has an iterate of finite trace class.

Section 6 is concerned with the skew-Hermitian part S_ξ of the resolvent of $(A_2 + \xi I)^{-1}$, ξ real. It establishes that if the coefficients of the skew-symmetric part of A go to zero at infinity, the essential spectrum of A_p and of H_p, where H is the symmetric part of A, coincide, while $(A_p + \xi I)^{-1} - (H_p + \xi I)^{-1}$ is a compact operator. Under stronger assumptions on the rate of decrease of the skew-symmetric coefficients at infinity, it is shown that S_ξ has an iterate of finite trace class, and more precisely that $(A_2 + \xi I)^{-s}$ for some sufficiently large s has a skew-Hermitian part of finite trace class. Finally, Section 7, as we have noted above, gives certain sharper conclusions for second-order operators by elementary arguments.

The writer is indebted for some stimulating conversations concerning the subject matter of Section 7 with Prof. K. Yosida. After this section was written, we have remarked that the same device which is applied in the first part of that section to second-order operators had already been used for a slightly different purpose by E. Gagliardo in [23]. The writer also wishes to acknowledge the assistance of some remarks by J. T. Schwartz in simplifying the discussion of the first part of Section 6, as noted in the argument of that section.

Section 1: Function spaces

Let G be an open set of the Euclidean n-dimensional space E^n, $n \geqq 1$. The points of E^n will be denoted by $x = (x_1, \ldots, x_n)$ and integration with respect to dx over a subset of E^n is integration with respect to Lebesgue n-measure. To write our differential operators, let $D_j = (i^{-1}\partial/\partial x_j)$ for $1 \leqq j \leqq n$. Then if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is any n-tuple of non-negative integers, we set $|\alpha| = \sum_{j=1}^{n} \alpha_j$, $D^\alpha = \prod_{j=1}^{n} D_j^{\alpha_j}$.

Any partial differential operator A of order $2m$ with coefficients defined on G may be written in the form

$$A = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha,$$

where the coefficients a_α are complex-valued functions on G. Throughout all

of our discussion we make the following regularity and boundedness assumptions on the coefficients a_α:

(I) *The coefficients a_α are measurable and uniformly bounded on G for all α.*

(II) *The top-order coefficients, i.e. a_α for $|\alpha| = 2m$, are uniformly continuous on G.*

The characteristic form $a(x, \xi)$ of A is defined for any point x of G and any real n-vector ξ by

$$a(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha, \quad \left(\xi^\alpha = \prod_{j=1}^{n} \xi_j^{\alpha_j} \right).$$

A is uniformly elliptic on G if there exists a constant $c_0 > 0$ such that $|a(x, \xi)| \geqq c_0 |\xi|^{2m}$ for every point x of G and any real n-vector ξ. A is uniformly strongly elliptic if there exists a constant $c_0 > 0$ such that $\mathrm{Re}\,(a(x, \xi)) \geqq c_0 |\xi|^{2m}$ for every such x and ξ. A is essentially real if the top-order coefficients of A, a_α for $|\alpha| = 2m$, are real-valued functions on G.

The formal adjoint A' of A is defined as a differential operator with distribution coefficients by

$$A'u = \sum_{|\alpha| \leqq 2m} D^\alpha(\bar{a}_\alpha(x))u = \sum_{|\alpha| \leqq 2m} a'_\alpha D^\alpha u\,,$$

for all u in $C_c^\infty(G)$, the family of infinitely differentiable functions with compact support in G. Most (though not all) of our results are obtained under the following hypothesis on the coefficients of A':

(III) *The coefficients a'_α of A' are measurable and essentially bounded functions on G.*

The regularity condition that we impose upon the open set G on which our differential operator is defined is given in the following definition:

Definition 1: *Let G be an open set of E^n with boundary Γ. Then G is said to be uniformly regular of class C^{2m} if there exists a family of open sets $\{N_k\}$ of E^n and of homeomorphisms $\{\Phi_k\}$ of N_k upon the unit ball $\{y : |y| < 1\}$ in E^n, and an integer R such that the following conditions are satisfied:*

(1) *Let $N'_k = \Phi_k^{-1}(|y| < 1/2)$. Then $\bigcup_k N'_k$ contains the R^{-1}-neighborhood of Γ in G.*

(2) *For each k, $\Phi_k(N_k \cap G) = \{y : y_1 > 0, |y| < 1\}$, while $\Phi_k(N_k \cap \Gamma) = \{y : y_1 = 0, |y| < 1\}$.*

(3) *Any $(R+1)$ distinct open sets $\{N_k\}$ have an empty intersection.*

(4) *Let Ψ_k be the mapping of $\{y : |y| < 1\}$ on N_k which is inverse to Φ_k. Then Φ_k and Ψ_k are mappings of class C^{2m}, and if Φ_{jk} and Ψ_{jk} are the j-th components of Φ_k and Ψ_k respectively, there exists a constant M (independent of x, y, or k) such that*
$|D^\beta \Phi_{jk}(x)| \leqq M$, $|D^\beta \Psi_{jk}(y)| \leqq M$, $\Phi_{1,k}(x) \leqq M \,\mathrm{dist}\,(x, \Gamma)$, *for* $|\beta| \leqq 2m$, *x in N_k, and $|y| < 1$.*

The simplest examples of domains G which are uniformly regular of class C^{2m} are the interiors or exteriors of bounded $(n-1)$-dimensional hypersurfaces which are locally of class C^{2m}, the whole of E^n, or a half-space bounded

by a hyperplane. Throughout most of our results, we assume, in addition to uniform regularity of class C^{2m} for G, that Γ is locally of class C^{4m}.

Definition 2: *Let p be a real number with $1 < p < \infty$, j a non-negative integer. Then $W^{j,p}(G)$ consists of those functions u in $L^p(G)$ for which the distribution derivative $D^\beta u$ also lies in $L^p(G)$ for $|\beta| \leqq j$. $W^{j,p}(G)$ is a Banach space under the (j, p)-norm,*

$$\|u\|_{j,p} = \left\{ \sum_{|\beta| \leqq j} \|D^\beta u\|^p_{L^p(G)} \right\}^{1/p}.$$

Since $1 < p < \infty$, $W^{j,p}(G)$ is a reflexive separable Banach space since it can be mapped isometrically on a closed subspace of a product of separable L^p spaces.

For $p = 2$, $W^{j,2}(G)$ is a Hilbert space with respect to the inner product

$$(u, v)_j = \sum_{|\beta| \leqq j} (D^\beta u, D^\beta v); \; u, v \in W^{j,2}(G) \,,$$

where $(u, v) = \int_G u(x)\, \bar{v}(x)\, dx$ is the ordinary L^2-inner product in G.

Lemma 1: *Let $R(G)$ be the family of functions on G which are the restrictions to G of functions in $C_c^\infty(E^n)$. Then if the boundary Γ of G is locally of class C^2, $R(G)$ is dense in $W^{j,p}(G)$ for every non-negative integer j and all p with $1 < p < \infty$.*

Proof of Lemma 1: Let u be an element in $W^{j,p}(G)$ for some j and p. We wish to show that u can be approximated arbitrarily closely in $W^{j,p}(G)$ by elements of $R(G)$. We first approximate u by an element v of $W^{j,p}(G)$ with bounded support, then v by a finite sum of elements with small support, and then each of the latter by elements of $R(G)$. If each of these approximations is arbitrarily fine, their combination will yield the result.

Let $\varepsilon > 0$ be given. Let $\lambda(x)$ be a function from $C_c^\infty(E^n)$ with $\lambda(x) = 1$ for $|x| \leqq 1$, $0 \leqq \lambda(x) \leqq 1$, $\lambda(x) = 0$ for $|x| \geqq 2$. For every positive integer k, let $\lambda_k(x) = \lambda(k^{-1}x)$. Then $\lambda_k(x) = 1$ for $|x| \leqq k$, 0 for $|x| \geqq 2k$, $D^\beta(\lambda_k(x)) = k^{-|\beta|}(D^\beta\lambda)\,(k^{-1}x)$. Consider $v_k = \lambda_k(x)u$. For each k, since u lies in $W^{j,p}(G)$, v_k must lie in $W^{j,p}(G)$. On the other hand, for each β with $|\beta| \leqq j$,

$$D^\beta(u - v_k) = D^\beta(\{u(1 - \lambda_k)\} = (1 - \lambda_k)D^\beta u + \sum_{|\alpha| < |\beta|} c_{\beta,\alpha}$$
$$k^{-(|\beta|-|\alpha|)}(D^{\alpha-\beta}\lambda)\,(k^{-1}x)\, D^\alpha u \,.$$

Since each function $D^{\alpha-\beta}\lambda$ is uniformly bounded on E^n, we obtain from this last equality that

$$\|D^\beta(u - v_k)\|_{L^p(G)} \leqq \|D^\beta u\|_{L^p(G_k)} + C k^{-1} \|u\|_{j-1,p} \,,$$

where G_k is the part of G where $|x| \geqq k$. Since each of the terms on the right hand side approach zero as $k \to \infty$, it follows by choosing k sufficiently large, we make $\|u - v_k\|_{j,p} < \varepsilon/3$.

Let v be such a v_k. The support of v is compact, and since the boundary Γ of G is locally of class C^2, there exists a bounded open subset G_0 of G whose boundary Γ_0 is the union of a finite number of non-intersecting $(n-1)$-dimensional manifolds of class C^2 in E^n and such that the support of v is a compact subset of $G_0 \cup (\Gamma \cap \Gamma_0)$. From the smoothness condition on the

boundary of G_0, it follows that we can find a finite open covering $\{N_1, \ldots, N_R\}$ of $cl(G_0)$ in E^n such that for each k, either N_k is completely contained in G_0 or there exists a unit vector ξ_k in E^n such that for all sufficiently small $t > 0$, the translation of $cl(N_k) \cap cl(G_0)$ by $t\xi_k$ is contained in G_0. Corresponding to this covering, it follows by a standard argument that there exists a family of functions $(\eta_1, \ldots, \eta_R)$, with each η_k in $C_c^\infty(E^n)$ and having its support in N_k, such that $\sum_{k=1}^{R} [\eta_k(x)]^2 = 1$ for all x in G_0.

Let $w_k = \eta_k v$. Then w_k lies in $W^{j,p}(N_k)$, and if we can find a function f_k from $C_c^\infty(E^n)$ such that $\|w_k - f_k\|_{W^{j,p}(N_k)} < (3KR)^{-1}$, where $K = 2^j j^{2n} \sup_k \|\eta_k\|_{C^j(N_k)}$, then setting $f = \sum_k \eta_k f_k$, we have f in $R(G)$ and,

$$\|(f - v)\|_{W^{j,p}(G_0)} \leqq \sum_k \|\eta_k f_k - \eta_k w_k\|_{W^{j,p}(N_k)} \leqq$$

$$\leqq RK \sup_k \|w_k - f_k\|_{W^{j,p}(N_k)} < \varepsilon/3 .$$

If $cl(N_k)$ lies in G_0, then w_k may be approximated by $j_c * w_k$ in $W^{j,p}(N_k)$ where j_c is a sequence of functions from $C_c^\infty(E^n)$ converging to the Dirac delta function in the distribution sense. More precisely, let j be a function from $C_c^\infty(E^n)$ such that $j(x) = 0$ for $|x| \geqq 1$, $\int j(x)\,dx = 1$, and set $j_c(x) = c^{-n} j(c^{-1}x)$ for $c > 0$. To handle the case where $cl(N_k)$ intersects Γ_0, let c and t be positive and small, and let

$$s_{c,t}(x) = \int_{G_0} j_c(x - y) w_k(y + t\xi_r)\,dy .$$

Obviously $s_{c,t}$ lies in $C_c^\infty(E^n)$, and our proof will be complete if we show that for suitable choices of c and t, $\|s_{c,t} - w_k\|_{W^{j,p}(N_k \cap G_0)}$ can be made arbitrarily small. For x in G_0 and $|\beta| \leqq j$, we have

$$(D^\beta s_{c,t})(x) = \int_{N_r \cap G_0} \{D_x^\beta j_c(x - y)\} w_k(y + t\xi_r)\,dy$$

$$= (-1)^{|\beta|} \int_{N_r \cap G_0} \{D_y^\beta j_c(x - y)\} w_k(y + t\xi_r)\,dy ,$$

provided that $t < \operatorname{dist}(\sup(w_r), E^n - N_k)$. By a change of variables, the last integral equals

$$(1.1) \qquad (-1)^{|\beta|} \int_{(G_0 \cap N_k) + t\xi_k} \{D_y^\beta j_c(x + t\xi_k - y)\} w_k(y)\,dy .$$

Suppose $t > 0$ is given. By our assumption on the covering $\{N_k\}$, the set $(G_0 \cap N_k) + t\xi_k$ has a positive distance $d_k(t)$ from the complement of G_0. If we take $c < d_k(t)$, then $j_c(x + t\xi_k - y)$ as a function of y will have its support in the c-neighborhood of $(G_0 \cap N_k) + t\xi_k$, i.e., it will have compact support in G_0. In that case, the integral (1.1) becomes

$$(1.2) \qquad \int_{(G_0 \cap N_k) + t\xi_k} j_c(x + t\xi_r - y)\,(D^\beta w_k)(y)\,dy ,$$

which in turn by a change of variables becomes,

$$(1.3) \qquad \int_{G_0 \cap N_k} j_c(x - y)\,(D^\beta w_k)(y + t\xi_k)\,dy .$$

As $t \to 0$, $(D^\beta w)(y + t\xi_k)$ converges to $(D^\beta w)(y)$ in $L^p(N_k \cap G_0)$. Suppose $\varepsilon_1 > 0$ is given. Choose $t > 0$ so that the $L^p(N_k \cap G_0)$-norm of their difference is $< \varepsilon_1/j^2$. On the other hand by a familiar calculation, for fixed t, $D^\beta s_{c,t}$ as expressed by (1.3) converges to $D^\beta w_k(y + t\xi_k)$ in $L^p(N_k \cap G_0)$ as $c \to 0$. Choosing first t and then c sufficiently small, we obtain

$$\|s_{c,t} - w_k\|_{W^{j,p}(N_k \cap G_0)} < \varepsilon_1 ,$$

and our conclusion is established.

If Γ is locally of class C^1, its imbedding in E^n induces a $(n-1)$-dimensional area element on Γ which we denote by dS. We define the space $L^p(\Gamma)$ with respect to the measure dS in the usual fashion. If Γ is locally of class C^{2m}, the class $C_c^{2m}(\Gamma)$ of $2m$-times continuously differentiable functions with compact supports in Γ is well-defined, and so is its dual, the space of distributions on Γ of order at most $2m$. If f is a locally summable function on Γ (with respect to dS) and ϱ is an element of $C_c^{2m}(\Gamma)$ whose support is contained in a coordinate patch with coordinates $(u_1, \ldots, u_{n-1})$, the distribution corresponding to f is given by $\lambda_f(\varrho) = \int f(x)\,\varrho(x)\,dS = \int_{P_{n-1}} f(x(u))\,\varrho(x(u))\,J(u)\,du$. For each $\beta = (\beta_1, \ldots, \beta_{n-1})$ with $|\beta| \leqq 2m - 1$, the tangential distribution derivative $\hat{D}^\beta f = f_\beta$ is defined in the coordinate patch by $\lambda_{f_\beta}(\varrho) = (-1)^{|\beta|} \int_{P_{n-1}} f(x(u)) \times$ $\times \hat{D}_u^\beta\{\varrho(x(u))\,J(u)\}\,du$. The function f is said to be locally in $W^{j,p}(\Gamma)$ if f_β is locally in $L^p(\Gamma)$ for every β with $|\beta| \leqq j$. To simplify our definitions, we only define $W^{j,p}(\Gamma)$ itself for the case when G is uniformly regular of class C^{2m} and $j < 2m$. In that case, if f is locally in $W^{j,p}(\Gamma)$ as defined above, consider f on $N_k \cap \Gamma$ for each neighborhood N_k of the regularity definition which intersects Γ. Let B_0 be the unit ball in E^n, P_0 its intersection with the hyperplane $y_1 = 0$. The mapping Φ_k maps $N_k \cap \Gamma$ onto P_0, and if Ψ_k is the inverse of Φ_k, $g_k(y) = f(\Psi_k(y))$ lies in $W^{j,p}(P_0)$, where P_0 is considered as an open subset of the Euclidean $(n-1)$-dimensional space. Then f lies in $W^{j,p}(\Gamma)$ if and only if $\sum_k \|g_k\|^p_{W^{j,p}(P_0)} < \infty$, and the $W^{j,p}(\Gamma)$-norm of f is defined by

$$\|f\|_{W^{j,p}(\Gamma)} = \left\{\sum_k \|g_k\|^p_{W^{j,p}(P_0)}\right\}^{1/p} .$$

Lemma 2: *Let G be an open set of E^n with boundary Γ, and suppose that G is uniformly regular of class C^{2m}. Then there exists an integer R and constants $K > 0$, $\delta_0 > 0$, such that given δ with $0 < \delta < \delta_0$, there exists a countable covering $\{N_k\}$ of $cl(G)$ by open subsets of E^n, a family of functions $\{\eta_k\}$ with $\eta_k \in C_c^{2m}(N_k)$, and a family of homeomorphisms $\{\Phi_k\}$ of N_k into E^n which satisfy the following conditions:*

(a) At most R distinct members of the covering $\{N_k\}$ have a non-empty intersection, while each of the N_k has diameter $< \delta$.

(b) For every k, the image of N_k under Φ_k is the ball of B_δ of radius δ about the origin. If N_k intersects the complement of G, $\Phi_k(N_k \cap G) = \{y : |y| < \delta,\ y_1 > 0\}$, $\Phi_k(N_k \cap \Gamma) = \{y : |y| < \delta,\ y_1 = 0\}$. In either case, if Ψ_k is the inverse mapping

to Φ_k and $\Phi_{r,k}$, $\Psi_{r,k}$ are the r-th components of Φ_k and Ψ_k respectively, we have

$$|D^\beta \Phi_{r,k}(x)| \leqq K\,, \quad |D^\beta \Psi_{r,k}(y)| \leqq K$$

for all β with $|\beta| \leqq 2m$, all x in N_k, and all y in B_δ.

(c) For every α and β with $|\alpha| \leqq 2m$, $|\beta| \leqq 2m$, we have

$$\sum_{k=1}^{\infty} |D^\alpha \eta_k(x)|^p \cdot |D^\beta \eta_k(x)|^p \leqq K^p \delta^{-(|\alpha|+|\beta|)p} < \infty,\ 1 < p < \infty.$$

(d) For every x in G, $0 \leqq \eta_k(x) \leqq 1$, and,

$$\sum_k [\eta_k(x)]^2 = 1\,.$$

Proof of Lemma 2: By the Definition 2 of the C^{2m}-uniform regularity of G, there exist a sequence of open sets $\{N_{k,0}\}$ of E^n and a corresponding sequence $\{\Phi_{k,0}\}$, with $\Phi_{k,0}$ a homeomorphism of $N_{k,0}$ on the unit ball in E^n, which satisfy the conditions of Definition 1. (We introduce the subscript " $,0$" to distinguish these open sets and homeomorphisms from those whose existence is asserted in Lemma 2, the latter depending, of course, upon the given constant δ.) In particular, there exists an integer R_0 such that not more than R_0 of the sets $\{N_{k,0}\}$ have a non-empty intersection, and such that, if $N'_{k,0} = \Phi_{k,0}^{-1}$ ($|y| < 1/2$), the sequence $\{N'_{k,0}\}$ forms a covering of the (R_0^{-1})-neighborhood of Γ in G. In addition, the bounds on the first derivatives of the components of $\Phi_{k,0}$ and its inverse $\Psi_{k,0}$ imply that the diameter of $N_{k,0}$ lies between K_0^{-1} and K_0 for all k, where K_0 is the constant bound in part 4 of Definition 1. If S is a part of the unit ball of diameter d, the image of S under $\Psi_{k,0}$ has diameter $\leqq Kd$. Moreover, if for $0 < d < (2R_0)^{-1} S_d$ is the subset of the ball of radius $1/2$ for which $|y_1| < d$ and $S_{d,k} = \Psi_{k,0}(S_d)$, then the union of the $S_{d,k}$ over all k contains the $(K^{-1}d)$-neighborhood of Γ in G. Indeed since $|\Phi_{r,0}(x)| \leqq K \times$ $\times$ dist(x, Γ) for all x in $N_{r,0}$, every point in the $(K^{-1}d)$-neighborhood of Γ is mapped by $\Phi_{k,0}$ into a point in the unit ball for which $|y_1| < d$, while every such point in the $(K^{-1}d)$-neighborhood of Γ is contained in the union of the $N'_{r,0}$.

As before, let us denote by B_0 the unit ball in E^n, P_0 its intersection with the hyperplane $y_1 = 0$, and let B_1 be the ball of radius $1/2$ about the origin in E^n, P_1 the intersection of B_1 with the plane $y_1 = 0$. Suppose that $d_1 > 0$, is given, with $d_1 < (2n^{1/2} + 2)^{-1}$. Consider the lattice of points in E^{n-1} having the form $d_1 \hat{y}_1$, where the components of the point $\hat{y}_I = (y_2, \ldots, y_n)$ are integers, and for each of the points of this lattice that lies in P_1 construct the ball in E^n with the center at this point and radius $(n^{1/2}+1)d_1$. Each of these balls is contained in B_0, their union contains all points of B_1 for which $|y| \leqq d_1$, and by our preceding remarks, the image under $\Psi_{k,0}$ of each of these balls is a set of diameter $< 2K_0(n^{1/2}+1)d_1$. Furthermore, the union over k of the images under $\Psi_{k,0}$ of this family of balls contains the set of all points in G whose distance from Γ is less than $K_0^{-1}d_1$.

Let $d_2 = (2K_0 n^{1/2})^{-1} d_1$, and consider the lattice of points in E^n having the form $d_2 x_1$, where x_1 has integer components. For each point of this lattice lying in G and having distance $\geqq K_0^{-1}d_1$ from Γ, construct the ball of radius

$n^{1/2}d_2$ with center at this point. Each of these balls is contained in G, and moreover the union of these balls, together with the images under the various $\Psi_{k,0}$ of the family of balls constructed in the last paragraph, covers G. This combined family of sets is the family $\{N_k\}$ sought in Lemma 2, provided that $2K_0(n^{1/2}+1)d_1 = \delta$. We remark that property (a) of Lemma 2 is verified, since all the sets have diameter $< \delta$, while the number which have a non-empty intersection is less than $2k(n)$, where $k(n)$ is the number of integer lattice points in E^n contained in the sphere of radius $(n^{1/2}+1)$ about the origin.

Property (b) is easily verified. For the N_k which intersect Γ, the mapping Φ_k is defined as the composition of $\Phi_{k,0}$ with a translation to the origin and an uniform dilation. For the sets N_k in the interior which are themselves balls, Φ_k is simply a translation followed by a dilation. The inequalities of (b) follow therefore from the corresponding inequalities for the mappings $\Phi_{k,0}$ and $\Psi_{k,0}$.

For the construction of the family of functions $\{\eta_k\}$, we proceed as follows: Each of the sets N_k has a mapping Φ_k on the ball of radius δ in E^n, and from our choice of constants it follows that $\bigcup_k \Phi_k^{-1}(|y| < \Theta)$ is a covering of G, where $\Theta < 1$ is a constant depending only on n. Let $\lambda(x)$ be a function from $C_c^\infty(E^n)$, $0 \leqq \lambda(x) \leqq 1$, having its support in the unit sphere and equal to 1 in the sphere $|y| \leqq \Theta$. We define $\lambda_k(x) = \lambda(\delta^{-1}\Psi_k(x))$. The functions λ_k lie in $C_c^{2m}(E^n)$ and have their supports in N_k for each k. Furthermore, $|D^\beta \lambda_k(x)| \leqq \leqq K_1 \delta^{-|\beta|}$ for $|\beta| \leqq 2m$.

Let $\eta_k(x) = \lambda_k(x)\left\{\sum_k [\lambda_k(x)]^2\right\}^{-1}$. The functions η_k lie in $C_c^{2m}(N_k)$, at most R of them are different from zero for any given x, and $\sum_k [\lambda_k(x)]^2$ being $\geqq 1$ for all x in G, we see immediately that $|D^\beta \eta_k(x)| \leqq K_2 \delta^{-|\beta|}$ for $|\beta| \leqq 2m$. Thus

$$\sum_k |D^\alpha \eta_k(x)|^p \cdot |D^\beta \eta_k(x)|^p \leqq K_2^{2p} R \delta^{-(|\alpha|+|\beta|)p}.$$

Finally it is obvious from the definition of the $\eta_k(x)$ that (d) must hold.

A useful consequence of Lemma 2 is the following:

Lemma 3: *Let $\{\eta_k\}$ be a family of functions satisfying the conditions of Lemma 2. Then there exist constants $K_j > 0$, $0 \leqq j \leqq 2m$, depending only on p and the constants K and R of Lemma 2, such that for all u in $W^{2m,p}(G)$ we have for $j \leqq 2m$,*

$$\|u\|_{j,p}^p \leqq K_j \sum_k \|\eta_k^2 u\|_{j,p}^p < \infty,$$

$$\sum_k \|\eta_k^2 u\|_{j,p}^p \leqq K_j \left\{\sum_{j_1 \leqq j} \delta^{-(j-j_1)p} \|u\|_{j_1,p}^p\right\} < \infty.$$

Proof of Lemma 3: We begin with the proof of the finiteness of the sum in k on the right hand side of the first inequality. Let S be a large positive integer and consider

$$\sum_{k=1}^S \|\eta_k^2 u\|_{j,p}^p = \sum_{k=1}^S \sum_{|\beta| \leqq j} \int_G |D^\beta(\eta_k^2(x)\, u(x))|^p \, dx$$

$$= \sum_{|\beta| \leqq 1} \sum_{k=1}^S \int_G \left| \sum_{\alpha,\lambda,\zeta} c_{\beta\alpha\zeta} D^\alpha(\eta_k) D^\lambda(\eta_k) D^\zeta u \right|^p dx,$$

(where in the final sum, $|\alpha| + |\lambda| + |\zeta| = |\beta| \leqq j$, and the constants $c_{\beta\alpha\zeta}$ depend only upon the indices.) Using the elementary inequality $\left|\sum_{k=1}^{T} a_k\right|^p \leqq$ $\leqq 2^{T(p-1)} \sum_{k=1}^{T} |a_k|^p$, together with part (c) of Lemma 2 and the fact that the indices of differentiation are bounded in number by a function of j, we obtain

$$\sum_{k=1}^{S} \|\eta_k^2 u\|_{j,p}^p \leqq \sum_{\beta,\alpha,\lambda,\zeta} K_k' \int_G \left\{ \sum_{k=1}^{S} |D^\alpha \eta_k|^p |D^\lambda \eta_k|^p |D^\zeta u|^p \right\} dx \leqq$$
$$\leqq K_k \sum_{j_1 \leqq j} \delta^{-(j-j_1)p} \|u\|_{j_1,p}^p ,$$

with the constant K_k independent of S. Letting $S \to \infty$, we see that the infinite sum is finite and that the second inequality holds.

To obtain the first inequality, consider for $|\beta| \leqq j$,

$$\int_G |D^\beta u|^p dx = \int_G \left| D^\beta \left(\sum_{k=1}^{\infty} [\eta_k]^2 u \right) \right|^p dx .$$

For each point x in C, there are at most R terms in the sum in the integrand of the last integral which differ from zero. Hence the integral is less than

$$2^R \sum_k \int_G |D^\beta(\eta_k^2 u)|^p dx \leqq 2^R \sum_k \|\eta_k^2 u\|_{j,p}^p$$

and summing on β, we obtain the first inequality.

The first application which we shall make of Lemma 3 is to study the existence of generalized boundary values for the derivatives of functions in $W^{2m,p}(G)$ and the norms of these boundary-value functions on Γ. Consider the differential operator D^β on E^n with $|\beta| < 2m$. If u lies in $R(G)$, (which we recall consists of restrictions of infinitely differentiable functions with bounded support), then the restriction of $D^\beta u$ to Γ (which we write in the usual fashion as $D^\beta u|_\Gamma$), is a smooth function with compact support on Γ. In particular, $D^\beta u|_\Gamma$ lies in $W^{2m-1-|\beta|,p}(\Gamma)$. To extend the definition of $D^\beta|_\Gamma$ to the whole of $W^{2m,p}(G)$, we apply the following result:

Lemma 4: *Suppose that G is an open set of E^n with boundary Γ and that G is uniformly regular of class C^{2m}. Let D^β be an elementary differential operator on E^n with $|\beta| < 2m$. Then there exists a constant C such that if $u \in W^{2m,p}(G)$, $v = D^\beta u|_\Gamma$, then*

$$\|v\|_{W^{2m-1-|\beta|,p}(\Gamma)} \leqq C \|u\|_{W^{2m,p}(G)} .$$

In particular $D^\beta|_\Gamma$ may be extended in a unique fashion by continuity to a bounded linear mapping of all of $W^{2m,p}(G)$ into $W^{2m-1-|\beta|,p}(\Gamma)$. (We shall continue to denote this mapping by $D^\beta|_\Gamma$.)

Proof of Lemma 4: By Lemma 3 and the definition of the norm in $W^{j,p}(\Gamma)$, it suffices to prove the inequality of Lemma 4 for $[\eta_k]^2 u$ instead of u, where η_k is one of the family of functions described in Lemma 2 for some fixed $\delta > 0$. Transplanting the function $\eta_k^2 u$ to the unit ball in E^n by the mapping Φ_k we may assume without any loss of generality that the function u which we are

considering has compact support in the intersection of the half-space $y_1 \geqq 0$ with the unit ball B_0 and that Γ is the intersection P_0 of B_0 with the plane $y_1 = 0$. An elementary integration by parts yields the integral representation

$$(D^\beta u)(\hat{y}) = \int_{B_0} \sum_{j=1}^{n} |x - \hat{y}|^{-n}(x_j - y_j)\,(iD_j D^\beta u)(x)\,dx\,,$$

for $\hat{y} \in P_0$, $y_1 = 0$. Thus

$$|D^\beta u(\hat{y})| \leqq \sum_{|\alpha| = |\beta|+1} \int_{B_0} |x - \hat{y}|^{1-n}\,|D^\alpha u(x)|\,dx\,, \quad (\hat{y} = (y_2, \ldots, y_n)).$$

Our conclusion will follow if we show that the integral transformation $(Tu)(\hat{y}) = \int_{B_0} |x - \hat{y}|^{1-n} u(x)\,dx$ is a bounded mapping of $L^p(B_0)$ into $L^p(P_0)$. To establish this latter fact, we note first

$$|x - \hat{y}|^{1-n} \leqq x_1^{-a}|x - \hat{y}|^{-(n-1-a)}, \quad a > 0\,.$$

Suppose $a < 1$ and let $w_{x_1}(\hat{y}) = \int_{P_0} |x - \hat{y}|^{-(n-1-a)}|u(x_1, x)|\,dx$. Then for $\hat{y}$ in P_0,

$$|Tu(\hat{y})| \leqq \int_0^1 x_1^{-a} w_{x_1}(\hat{y})\,dx_1.$$

Let $q \geqq 1$. Then by the usual estimation for Riemann integrals for functions with values in a Banach space,

$$\|Tu\|_{L^q(P_0)} \leqq \int_0^1 \|w_{x_1}\|_{L^q(P_0)}\, x_1^{-a}\,dx_1\,. \tag{1.4}$$

By a well-known theorem of Sobolev [65], (a simple reduction of which to the corresponding one-dimensional fractional integral theorem of Hardy and Littlewood is given by DuPlessis in [24]), we have the inequality

$$\|w_{x_1}\|_{L^q(P_0)} \leqq C_a \left\{\int_{P_0} |u(x_1, \hat{x})|^p\,d\hat{x}\right\}^{1/p}, \tag{1.5}$$

provided that $p^{-1} - a(n-1)^{-1} \leqq q^{-1} \leqq p^{-1}$. Substituting (1.5) into the right side of (1.4) and applying Hölder's inequality, we obtain for such q,

$$\|Tu\|_{L^q(P_0)} \leqq C_a \left\{\int_0^1 \int_{P_0} |u(x_1, x)|^p\,dx\,dx_1\right\}^{1/p} \left\{\int_0^1 x_1^{ap/p-1}\,dx_1\right\}^{\frac{p-1}{p}}$$

If a is less than $(1 - p^{-1})$, we obtain

$$\|Tu\|_{L^q(P_0)} \leqq C_a'\,\|u\|_{L^p(B_0)}\,.$$

In particular, we have such an inequality for $q^{-1} > p^{-1} - (p-1)(pn-p)^{-1}$, and hence for $q = p$.

Remark: The proof of Lemma 4 is a variant of the proof of the Sobolev imbedding theorem ([65]) for the spaces $W^{j,p}(G)$. The corresponding result on the boundary (which we shall consider in its full generality in the second paper of this series) is like the imbedding theorem in that the top exponent is

actually attained. We need some special refinements of the Sobolev theorem in the present discussion, and in Lemma 5 which follows, we give a brief proof of these refinements.

Lemma 5: *(a) Let G be a domain satisfying the following cone condition: There exists constants $h, a > 0$ such that for each point x in G, there exists a measurable set S_x on the unit sphere $|\xi| = 1$ of the form $\{\xi : |\xi - \xi_x| \leqq a\}$ such that the cone $C_x = \{x + y \mid y = r\xi, r \leqq h, \xi \in S_x\}$ is contained in G and the distance from C_x to Γ is a continuous function of x in G.*

Then $W^{2m,p}(G) \subset W^{j,r}(G)$ for any r for which $(p^{-1} - (2m-j)n^{-1} \leqq r^{-1} \leqq p^{-1}$, and there exists a constant $C > 0$ (depending only on p, r, s, G and m) such that for all u in $W^{2m,p}(G)$, $j \leqq 2m$, r satisfying the above inequality and $s \leqq r$, we have

$$\|u\|_{W^{j,r}(G)} \leqq C \left\{ \int \sum_{|\alpha| = 2m} \|D^\alpha u\|^p_{L^p(G)} \right\}^{1/p} + \|u\|_{L^s(G)} \Big\} . \tag{1.6}$$

If $(p^{-1} - (2m-j)n^{-1}) < 0$, then $W^{2m,p}(G)$ lies in $C^j(\overline{G})$, the latter taken with the uniform norm on all derivatives of order $\leqq j$, and

$$\|u\|_{C^j(\overline{G})} \leqq C \|u\|_{W^{2m,p}(G)} .$$

(b) Suppose that in addition to the hypotheses of (a) there exists a covering of G by open sets $\{N_s\}$ satisfying the conditions of Definition 1 and such that for each N_s, there exists a mapping $\xi_s(x)$ of N_s into the unit sphere S^{n-1} for which:

(1) For each x in N_s, we may choose the sets S_x of (a) as $\{\xi : |\xi - \xi_s(x)| \leqq a\}$.

(2) There exists a constant K independent of S, such that $|\xi_s(x) - \xi_s(x_1)| \leqq K|x - x_1|$ for $x, x_1 \in N_s$.

Then every bounded set in $W^{2m,p}(G)$ satisfies a Hölder condition in $W^{j,r}(G)$ for $j < 2m$, $(p^{-1} - (2m-j)n^{-1}) < r^{-1} \leqq p^{-1}$, in the following sense: If $\lambda = n\{r^{-1} - p^{-1} + (2m-j)n^{-1}\}$, there exists $K_r > 0$ such that if $t \in E^n$ with $|t| \leqq d$, $G_t = G + t$, $v(x) = u(x - t)$ for x in G, then

$$\|v - u\|_{W^{j,r}(G \cap G_t)} \leqq K_r d^\lambda \|u\|_{W^{2m,p}(G)} .$$

In particular if G_1 is a bounded subset of G then every bounded set of $W^{2m,p}(G)$ is pre-compact in $W^{j,r}(G_1)$ for $(p^{-1} - (2m-j)n^{-1}) < r^{-1}$.

(c) Suppose that G satisfies the conditions of (b) and that $p^{-1} < (2m-j)n^{-1}$. Then $D^\alpha u$ will satisfy an uniform Hölder condition of order λ_1 on G for $|\alpha| \leqq j$ and $\lambda_1 < (2m-j) - np^{-1}$.

Remark: The hypotheses which we have made upon the open set G are un-necessarily stringent (as will be partly obvious from the proof given). In particular, $W^{2m,p}(G) \subset W^{j,p}(G)$ with the obvious norm inequality if the set S_x on the unit ξ-sphere is any measurable set whose measure is $\geqq a_1 > 0$, with a_1 independent of x. The conclusions of (b) and (c) of Lemma 5 are valid if G satisfies the strong cone condition of the Appendix of NIRENBERG [53], a hypothesis which is implied by our conditions on G. We refer for the proof of (c) to the latter paper.

Proof of Lemma 5(a). Let $j_k(x)$ be a sequence of functions from $C_c^\infty(E^n)$, as in the proof of Lemma 1, converging to the Dirac delta function, $u_k = j_k * u$, (with u extended to be null outside of G). Then $u_k \to u$ in $W^{2m,p}(G_1)$ as $k \to \infty$, where G_1 is any sub-domain of G whose distance from Γ is positive, and by choosing a subsequence, we may assume that $u_k \to u$ almost everywhere in G, $D^\alpha u_k \to D^\alpha u$, a.e. in G, where the u_k lie in $C^\infty(E^n)$. Let $g(r)$ be a function in $C_c^\infty(E^1)$ with $g(r) = 0$ for $r \geqq h$, $g(r) = 1$ for $r \leqq h/2$. For any point x in G, any ξ in S_x, and any v in $C^\infty(E^n)$

$$v(x) = -\int_0^h \partial/\partial r(g(r)\, v(x + r\xi))\, dr = \int_0^h \{g(r) \frac{\partial v}{\partial r}(x + r\xi) + \tag{1.7}$$
$$+ g'(r)\, v(x + r\xi)\}\, dr\,.$$

Let $f(\xi)$ be an infinitely differentiable function on the sphere $|\xi| = 1$ with compact support in the set $|\xi| < a$, such that $\int f(\xi)\, dS_\xi = 1$. Multiplying (1.7) by $f(\xi - \xi_s(x))$ for x in N_s and integrating with respect to dS over $|\xi| = 1$, we obtain

$$v(x) = -\int_{|y| \leqq h} \left\{ g(|y|) \sum_{j=1}^n i\, y_j |y|^{-1} (D_j v)(x + y) + g'(|y|)\, v(x + y) \right\} \times \tag{1.8}$$
$$\times |y|^{1-n} f(|y|^{-1} y - \xi_s(x))\, dy\,, \quad x \in N_s\,.$$

Since $g'(r) = 0$ for $2r \leqq h$, the function $\sigma_x(y) = g'(|y|)\, |y|^{1-n} f(|y|^{-1} y - \xi_s(x))$, of y lies in $C_c^\infty(E^n)$. Consider β with $0 < |\beta| < 2m$. Setting $v = D^\beta u_k$ in (1.8), we obtain

$$D^\beta u_k(x) = -\int g(|y|) \left\{ \sum_{j=1}^n i\, y_j |y|^{-n} (D_j D^\beta u_k)(x + y) \right\} f(|y|^{-1} y - \tag{1.9}$$
$$- \xi_s(x))\, dy - \int_{E^n} \sigma_x(y)\, (D^\beta u_k)(x + y)\, dy\,, \quad x \in N_s\,.$$

Transforming the second integral in (1.9) by integration by parts we obtain

$$(-1)^{|\beta|} \int_{E^n} (D^\beta \sigma_x)(y)\, (u_k)(x + y)\, dy\,, \tag{1.10}$$

where $(D^\beta \sigma_x)(y)$ has compact support in the set $\{y : y = r\xi,\ r \leqq h,\ \xi \in S_x\}$ and is uniformly bounded over all x in N_s and all s.

Substituting (1.10) into (1.9), we have

$$|D^\beta u_k(x)| \leqq C_0 \left\{ \int_{C_x} |y|^{1-n} \sum_{|\alpha| = |\beta| + 1} |D^\alpha u_k(x + y)|\, dy + \int_{C_x} |u(x + y)|\, dy \right\}. \tag{1.11}$$

Let G_1 be a bounded sub-domain of G with a positive distance from Γ. Since the distance of C_x from Γ is bounded from below on G_1, the union of the C_x for x in G_1 is contained in another such sub-domain G_2. If for the moment we set $u_k = 0$ outside of G_2, and remark that the integral transformations T and T_1 defined by

$$Tw(x) = \int_{|y| \leqq h} |y|^{1-n} w(x + y)\, dy\,,$$
$$T_1 w(x) = \int_{|y| \leqq h} w(x + y)\, dy\,,$$

both are bounded linear mappings of $L^p(E^n)\,(L^s(E^n))$ into $L^q(E^n)$, for $p^{-1} - n^{-1} \leqq q^{-1} \leqq p^{-1}$, $(s \leqq q)$ (the first by the Lemma of Sobolev), then, taking the limit on an infinite subsequence of the u_k, we obtain

$$|D^\beta u(x)| \leqq C_0 \left\{ \int_{C_x} |y|^{1-n} \sum_{|\alpha| = |\beta|+1} |D^\alpha u(x+y)|\, dy + \int_{C_x} |u(x+y)|\, dy \right\} \leqq$$
$$\leqq C \left\{ \sum_{|\alpha| = |\beta|+1} |T(D^\alpha u)(x)| + |T_1(u)(x)| \right\}, \tag{1.12}$$

for almost all in G_1. Hence,

$$\|D^\beta u\|_{L^q(G_1)} \leqq C_1 \left\{ \sum_{|\alpha| = |\beta|+1} \|D^\alpha u\|_{L^p(G)} + \|u\|_{S^q(G)} \right\}, \quad (s \leqq q) \tag{1.13}$$

with a constant C_1 independent of G_1 and s. Taking the limit as G_1 expands to fill out G and iterating on the order of β, we obtain (1.6). [The conclusion that $W^{2m,p}(G) \subset W^{j,r}(G)$ follows in the usual way from the remark that each u in $W^{2m,p}(G)$ is the limit in $W^{2m,p}(G_1)$ of the sequence u_k with $\|u_k\|_{W^{2,r}(G_1)}$ uniformly bounded.]

Proof of Lemma 5(b). Since the integrals on the right hand side of equation (1.9) are bounded mappings of $L^p(E^n)$ into $L^q(E^n)$ for the appropriate p and q, by taking the limit of an infinite subsequence of the u_k, we obtain the relation

$$D^\beta u(x) = \int_{E^n} g(|y|) \sum_{j=1}^{n} i y_j |y|^{-n} (D_j D^\beta u)(y+x)\, f(|y|^{-1} y - \xi_s(x))\, dy -$$
$$- \int_{E^n} \sigma_x(y)\, D^\beta u(x+y)\, dy, \tag{1.14}$$

for almost all x in N_s. Suppose that (1.14) holds for both x and $x_1 = x - t$, as it will on the complement of a null set in $N_s \cap G \cap G_t$. Then, we will have

$$D^\beta u(x) - D^\beta u(x_1) = \int_{E^n} \sum_j \{k_j(x,y) - k_j(x_1,y)\}\, (D_j D^\beta u)(y)\, dy +$$
$$+ \int_{E^n} \{k(x,y) - k(x_1,y)\}\, D^\beta u(y)\, dy, \tag{1.15}$$

with

$$k_j(x,y) = i g(|y-x|)\,(y_j - x_j)\,|y-x|^{-n} f(|y-x|^{-1}(y-x) - \xi_s(x)),$$
$$k(x,y) = \sigma_x(y-x), \tag{1.16}$$

respectively. Let $\lambda = n(r^{-1} - p^{-1} + n^{-1})$, where for the latter quantity we assume $0 < \lambda < 1$. A simple calculation yields a majoration of the form

$$|k_j(x,y) - k_j(x_1,y)| \leqq c\,|t|^\lambda \{|x-y|^{1-n-\lambda} + |x_1 - y|^{1-n-\lambda}\}. \tag{1.17}$$

Letting T_λ be the integral operator over $L^p(E^n)$ with kernel $|x-y|^{1-n-\lambda}$, it follows that T is a bounded mapping of L^p into L^r, and

$$\|D^\beta u - D^\beta u_t\|_{L^r(G \cap G_t)} \leqq c_1 |t|^\lambda \|u\|_{W^{1,p}(G \cap G_t)}. \tag{1.18}$$

Combining this inequaity with 5(a), we obtain the inequality of 5(b).

Let G_1 be a bounded subdomain of G, and for $\delta > 0$, let G_δ be the subset of G consisting of points at distance $> \delta$ from Γ. Then $G_{1,\delta} = G_1 \cap G_\delta$ is

increasing with decreasing δ and its limit is the whole of G_1. Let $G'_{1,\delta} = G_1 - (G_1 \cap G_\delta)$, so that $m(G'_{1,\delta}) \to 0$ as $\delta \to 0$. Let S be a bounded set in $W^{2m,p}(G)$, and suppose that $p^{-1} - (2m-j)n^{-1} = r_0^{-1} < r^{-1}$. Suppose that $\|u\|_{W^{2m,p}(G)} \leqq M_0$ for u is S. Then by Lemma 5(a), and Hölder's inequality

$$\int_{G'_{1,\delta}} |D^\beta u|^r \, dx \leqq \left\{ \int_{G'_{1,\delta}} |D^\beta u|^{r_0} \, dx \right\}^{r/r_0} \{m(G'_{1,\delta})^{(1-r/r_0)}\} \leqq \tag{1.19}$$

$$\leqq (cM_0)^{r/r_0} \, m(G'_{1,\delta})^{(1-r/r_0)}.$$

To prove the pre-compactness of S in $W^{j,r}(G)$, it suffices to prove the total boundedness of S in that space. Suppose we are given $\varepsilon > 0$. Then, since $m(G'_{1,\delta}) \to 0$ as $\delta \to 0$, there exists $\delta > 0$ so that by (1.19)

$$\|u\|_{W^{j,r}(G'_{1,\delta})} < 6^{-1}\varepsilon \tag{1.20}$$

for all u in S. Let η be any positive constant $< \delta$, and let $j_\eta(x)$ be one of our smoothing functions with support of diameter less than η. For each u in S, consider $u_\eta(x) = \int_G j_\eta(x-y)\, u(y)\, dy$, $x \in G_{1,\delta}$. If $|\beta| \leqq j$, $D^\beta u_\eta(x) = \int_G j_\eta(x-y)\, D^\beta u(y)\, dy$, since for x in $G_{1,\delta}$, the function $j_\eta(x-y)$ as a function of y has compact support in G. Therefore,

$$\|u_\eta - u\|_{W^{j,p}(G_{1,\delta})} = \left\| \int_G j_\eta(y) \{u(x+y) - u(x)\} \, dy \right\|_{W^{j,p}(G_{1,\delta})} \tag{1.21}$$

$$< \sup_{|y| \leqq \eta} \|u(x+y) - u(x)\|_{W^{j,p}(G \cap G_\delta)} < c\,\eta^\lambda,$$

where λ is the constant of the inequality 5(b) which we have already established. Thus, by choosing η sufficiently small independently of u in S, we can make $\|u_\eta - u\|_{W^{j,p}(G_{1,\delta})} < 6^{-1}\varepsilon$. Finally, the functions u_η, for fixed $\eta > 0$ as u ranges over S, are a uniformly bounded family in $C^\infty(G_{1,\delta})$ and we can find a finite subset of them which is $(3^{-1}\varepsilon)$-dense among them in the $W^{j,p}(G_{1,\delta})$-norm. The corresponding family of elements u of S is ε-dense in S in the $W^{j,p}(G_1)$-norm, since for any v in S,

$$\|u - v\|_{W^{j,p}(G_1)} \leqq \|u\|_{W^{j,p}(G'_{1,\delta})} + \|v\|_{W^{j,p}(G'_{1,\delta})} + \|u - v\|_{W^{j,p}(G'_{1,\delta})} < \varepsilon\,. \tag{1.22}$$

In the case that the domain G satisfies stronger regularity conditions, and in particular if it is uniformly regular of class C^{2m}, we can establish still more precise relations between the $W^{j,r}$ and $W^{2m,p}$ norms. To illustrate their precision, we begin by establishing a corollary of Lemma 5(b):

Corollary to Lemma 5. *Suppose that G satisfies the conditions of Lemma 5(b) and that G is bounded. Then given $\varepsilon > 0$, there exists $K_r(\varepsilon) > 0$ so that for all u in $W^{2m,p}(G)$, and $p^{-1} - (2m-j)n^{-1} < r^{-1}$,*

$$\|u\|_{W^{j,r}(G)} \leqq \varepsilon \|u\|_{W^{2m,p}(G)} + K_r(\varepsilon)\|u\|_{L^1(G)}. \tag{1.23}$$

Proof of the Corollary. Suppose the assertion false. Then for some $\varepsilon > 0$ and every integer k, there would exist non-zero u_k in $W^{2m,p}(G)$ such that

$$\|u\|_{W^{j,r}(G)} \geqq \varepsilon \|u\|_{W^{2m,p}(G)} + k\|u_k\|_{L^1(G)}. \tag{1.24}$$

Since (1.24) is homogeneous, we may assume that $\|u_k\|_{W^{2m,p}(G)} = 1$. By Lemma 5(b), the bounded sequence $\{u_k\}$ in $W^{2m,p}(G)$ has an infinite subsequence converging in $W^{j,r}(G)$, which without loss of generality we may take to be the original sequence. Let $u_k \to u_0$ in $W^{j,r}(G)$. By (1.24), we have $\|u_k\|_{W^{j,r}(G)} \geqq \varepsilon + k\|u_k\|_{L^1(G)}$, so that $u_k \to 0$ in $L^1(G)$. But since limits in $W^{j,r}(G)$ are also limits in $L^1(G)$, and since limits in $L^1(G)$ are unique, $u_0 = 0$. But,

$$\|u_0\|_{W^{j,r}(G)} = \lim \|u_k\|_{W^{j,r}(G)} \geqq \varepsilon > 0$$

by (1.24), so that we have a contradiction. Q. E. D.

Lemma 6. *Suppose that G is an open subset of E^n which is uniformly regular of class C^{2m} (but not necessarily bounded). Then for*

$$0 \leqq j < 2m,\ p^{-1} - (2m-j)n^{-1} < r^{-1} \leqq p^{-1};\ s \leqq r\ ,$$

there exists a constant $K(j, 2m, s, r, G) > 0$ such that for all u in $W^{2m,p}(G)$,

$$\|u\|_{W^{j,r}(G)} \leqq K\|u\|^{\lambda}_{W^{2m,p}(G)} \cdot \|u\|^{(1-\lambda)}_{L^q(G)} \tag{1.25}$$

where $\lambda = \dfrac{(ns - jrs - rn)}{(ns - jrs - rn + \varrho)}$ and $\varrho = r^{-1} - p^{-1} + (2m-j)n^{-1}$.

Moreover, there exists a constant K_1, depending only on K, such that for every $\varepsilon > 0$,

$$\|u\|_{W^{j,r}(G)} \leqq \varepsilon\|u\|_{W^{2m,p}(G)} + K_1\varepsilon^{-\mu}\|u\|_{L^q(G)}\ , \tag{1.26}$$

for all u in $W^{2m,p}(G)$ with $\mu = \varrho^{-1}(r^{-1} - j^{-1} - s^{-1})$, ϱ as above.

In particular, if $r = s = p$, we have

$$\|u\|_{W^{j,p}(G)} \leqq K\|u\|^{j/2m}_{W^{2m,p}(G)}\|u\|^{(2m-j)/2m}_{W^{2m,p}(G)}\ . \tag{1.27}$$

Proof of Lemma 6. We begin with a general remark about the relation between inequalities of the type of (1.25) and (1.26). Let a, b, and c be three positive numbers. If

$$a \leqq Kb^{\lambda}c^{1-\lambda} \tag{1.28}$$

where $0 < \lambda < 1$, and if ε is any positive number, then $a \leqq (b\varepsilon)^{\lambda}(b^{-\lambda}Kc^{1-\lambda})$, and applying the classical inequality $fg \leqq p^{-1}f^p + q^{-1}g^q$ for $p^{-1} + q^{-1} = 1$, we obtain,

$$a \leqq \varepsilon b + K_1\varepsilon^{-\mu}c\ , \tag{1.29}$$

with $\mu = \lambda(1-\lambda)^{-1}$ and K_1 a constant depending only on K and λ. Conversely, suppose that (1.29) holds for all $\varepsilon > 0$ with a constant K_1 independent of ε. Then, setting $\varepsilon = b^{\lambda-1}c^{+1-\lambda}$ in (1.29), we obtain inequality (1.28) with the constant $K = K_1 + 1$. The constants λ and μ in the inequalities (1.25) and (1.26) are related by $\mu = \lambda(1-\lambda)^{-1}$, so that the two inequalities are equivalent. We shall establish (1.26) in the following discussion.

Consider for a fixed $\delta > 0$, the family of functions $\{\eta_k\}$ whose existence was established in Lemma 2. By Lemma 3, for every u in $W^{j,r}(G)$, and a fortiori for u in $W^{2m,p}(G)$,

$$\|u\|^r_{W^{j,r}(G)} \leqq K_1\left\{\sum_k \|\eta_k^2 u\|^r_{W^{j,r}(G)}\right\}\cdot$$

By Lemma 2, moreover, the mapping $v_k \to w_k$ which assigns to a function v_k in $W^{j_k, r_1}(N_k \cap G)$, the corresponding function $w_k(y) = v_k(\psi_k(y))$ on either the unit ball B_0, or its intersection B_0^+ with the half-space $y_1 > 0$, is a bounded linear invertible mapping H_j of $W^{j_1, r_1}(N_k \cap G)$ into $W^{j_1, r_1}(B_0)$, $(W^{j_1, r_1}(B_0^+))$ for $j \leqq 2m$ and any r_1, with $\|H_k\|$, $\|H_k^{-1}\|$ bounded uniformly in k. If we wish to prove the inequality

$$\|\eta_k^2 u\|_{W^{j,r}(G)} \leqq \varepsilon \|\eta_k^2 u\|_{W^{2m,p}(G)} + K_1 \varepsilon^{-\mu} \|\eta_k^2 u\|_{L^p(G)}, \tag{1.30}$$

it suffices to prove the parallel inequality for the function $H_k(\eta_k^2 u)$.

For the latter proof, there are two cases, that of B_0 and of B_0^+. The proofs for the two cases are completely parallel, and we give the details only for the apparently more difficult case of B_0^+. The half-space $\{y : y_1 > 0\}$ is uniformly regular of class C^{2m} for every $m \geqq 1$, and applying inequality (1.6) of Lemma 5(a) to this domain G_0, we have for v in $W^{2m,p}(G_0)$,

$$\|v\|_{W^{j,r}(G_0)} \leqq C \left\{ \left(\sum_{|\alpha| = 2m} \|D^\alpha v\|_{L^p(G_0)}^p \right)^{1/p} + \|v\|_{L^p(G_0)} \right\}. \tag{1.31}$$

Let γ be a positive number. If u lies in $W^{2m,p}(B_0^+)$, then the function $v(y) = u(\gamma y)$ lies in $W^{2m,p}(G_0)$. Applying (1.31) to this function v and changing variables in the resulting integrals, we obtain, if $\gamma \leqq K_2$,

$$\gamma^{(j-nr^{-1})} \|u\|_{j,r} \leqq C \gamma^{(2m-np^{-1})} \|u\|_{2m,p} + \gamma^{-(ns^{-1})} \|u\|_{0,s}. \tag{1.32}$$

Dividing (1.32) by the power of γ on the left and substituting $\gamma = (C^{-1}\varepsilon)^{(n\varrho)^{-1}}$ with $\varrho = r^{-1} - p^{-1} + (2m - j)n^{-1}$, we obtain

$$\|u\|_{j,r} \leqq \varepsilon \|u\|_{2m,p} + C_1 \varepsilon^{-\mu} \|u\|_{0,s}, \tag{1.33}$$

with $\mu = \varrho^{-1}(r^{-1} - n^{-1}j - s^{-1})$, for $\varepsilon \leqq K_3$. For large ε, (1.33) follows from Lemma 5(a), so that (1.33) is true for all ε, and by our previous remarks, so is (1.30).

To complete the proof, we must show that (1.30) implies (1.26). But,

$$\|u\|_{j,r}^r \leqq K_1 \left\{ \sum_k \|\eta_k^2 u\|_{j,r}^r \right\} \leqq K_1 \varepsilon \left\{ \sum_k \|\eta_k^2 u\|_{2m,p}^r \right\} + C_2 \varepsilon^{-\mu} \left\{ \sum_k \|\eta_k^2 u\|_{0,s}^r \right\}.$$

Since $r \geqq p, s$, we have $\left(\sum_k \|\eta_k^2 u\|_{2m,p}^r \right) \leqq \left(\sum_k \|\eta_k^2 u\|_{2m,p}^p \right)^{r/p} \leqq K_3 \|u\|_{2m,p}^r$, the last by Lemma 3, and $\left(\sum_k \|\eta_k^2 u\|_{0,s}^r \right) \leqq \left(\sum_k \|\eta_k^2 u\|_{0,s}^s \right) \leqq K_r \|u\|_{0,s}^r$. Thus

$$\|u\|_{j,r}^r \leqq K_5 \varepsilon \|u\|_{2m,p} + K_5 \varepsilon^{-\mu} \|u\|_{0,s}^r, \tag{1.34}$$

for any $\varepsilon > 0$, and (1.34) is easily seen to be equivalent with (1.26).

The final result of this Section is the following variant of Lemma 6 which is applied in a localization argument in Section 2.

Lemma 7. *Let N_0 be the unit cube in E^n, $N_0 = \{y : |y_j| < 1$ for $1 \leqq j \leqq n\}$, N_0^+ the intersection of N_0 with the half-space $E_n^+ = \{y : y_1 > 0\}$. Let $\zeta(x) = \prod_{j=1}^{n} (1 - y_j^2)$ in N_0 and be zero outside N_0. Then for u in $W^{2m,p}(N_0)$, $\zeta^{2m} u$*

lies in $W^{2m,p}(E^n)$, and for each j with $0 < j < 2m$, we have an inequality of the form

$$\sum_{|\beta|=j} \|\zeta^j D^\beta u\|_{0,p}^p \leqq K \left\{ \sum_{|\beta| \leqq 2m} \|\zeta^{|\beta|} D^\beta u\|_{0,p}^p \right\}^{j/2m} \cdot \|u\|_{0,p}^{\frac{p(2m-j)}{2m}} \tag{1.35}$$

with a constant K independent of u.

Similarly if u lies in $W^{2m,p}(N_0^+)$, then $\zeta^{2m} u$ lies in $W^{2m,p}(E_n^+)$ and (1.35) *holds with the norms being taken over N_0^+.*

Proof of Lemma 7. It suffices by an inductive argument to prove for $0 < |\beta| = j < 2m$, that we have

$$\|\zeta^{|\beta|} D^\beta u\|_{0,p}^{2p} \leqq K \left\{ \sum_{|\alpha| \leqq j+1} \|\zeta^{|\beta|} D^\alpha u\|_{0,p}^p \right\} \cdot \left\{ \sum_{|\alpha| \leqq j-1} \|\zeta^{|\alpha|} D^\alpha u\|_{0,p}^p \right\}. \tag{1.36}$$

Since the proof given for Lemma 1 is also valid for any polyhedron in E^n, we may assume in the proof of (1.36), (since $R(N_0)$ is dense in $W^{2m,d}(N_0)$ and $R(N_0^+)$ is dense in $W^{2m,p}(N_0^+)$), that u is as smooth as we please. By the argument applied in the proof of Lemma 6, (1.36) is equivalent to the validity for every $\varepsilon > 0$ of the inequality

$$\begin{aligned} \|\zeta^{|\beta|} D^\beta u\|_{0,p}^p \leqq \varepsilon \sum_{|\alpha| \leqq j+1} \|\zeta^{|\alpha|} D^\alpha u\|_{0,p}^p + \\ + K\varepsilon^{-1} \sum_{|\alpha| \leqq j-1} \|\zeta^{|\alpha|} D^\alpha u\|_{0,p}^p . \end{aligned} \tag{1.37}$$

Since ζ is a product of functions of the separate variables, it suffices to prove for a function v of a single real variable s for $0 \leqq s \leqq 1$ that, setting $D_s = d/ds$,

$$\int_0^1 (1-s^2)^{jp} |D_s v|^p ds \leqq \varepsilon \int_0^1 (1-s^2)^{2jp} |D_s^2 v|^p ds + K\varepsilon^{-1} \int_0^1 |v|^p ds . \tag{1.38}$$

Indeed if x_k is a variable for which $\beta_k \neq 0$, and if we let β' equal β with the k-th index lowered by 1, then $D_k D^{\beta'} u = D^\beta u$, and setting $v(x_k) = D^{\beta'} u(x_k, \hat{x}_k)$, where $\hat{x}_k = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$, we obtain from (1.38),

$$\begin{aligned} \int_0^1 (1-x_k^2)^{jp} |D^\beta u(x_k, \hat{x}_k)|^p dx_k \leqq \varepsilon \int_0^1 (1-x_k^2)^{2jp} |D_k D^\beta u(x_k, \hat{x}_k)|^p dx_k + \\ + K\varepsilon^{-1} \int_0^1 |v(x_k, \hat{x}_k)|^p dx_k . \end{aligned} \tag{1.39}$$

Multiplying (1.39) by $\zeta(x)^{jp}(1-x_k^2)^{-jp}$, which is a function of $\hat{x}_k$, and setting $\varepsilon = \varepsilon_1 \cdot (\zeta(x))^{jp}(1-x_k^2)^{-jp}$, we obtain

$$\begin{aligned} \int_0^1 \{\zeta(x)\}^{jp} |D^\beta u(x_k, \hat{x}_k)|^p dx \leqq \varepsilon_1 \int_0^1 (\zeta(x))^{2jp} |D_k D^\beta u(x_k, \hat{x}_k)|^p dx_k + \\ + K\varepsilon_1^{-1} \int_0^1 |v(x_k, \hat{x}_k)|^p dx_k . \end{aligned} \tag{1.40}$$

Integrating (1.40) with respect to $d\hat{x}_k$ and applying the obvious majorations, we obtain (1.37) for N_0^+. Since (1.37) for N_0 obviously follows by adding (1.37) for N_0^+ and N_0^-, (1.38) obviously suffices to prove the Lemma.

We derive (1.38) from the pair of mutually equivalent weaker inequalities,

$$\int_a^b |D_s v|^p ds \leqq \varepsilon \int_a^b |D_s^2 v|^p ds + K\varepsilon^{-1} \int_a^b |v|^p ds\,, \tag{1.41}$$

$$\int_a^b |D_s v|^p ds \leqq K \left\{\int_a^b |D_s^2 v|\, ds\right\}^{1/2} \left\{\int_a^b |v|^p ds\right\}^{1/2}, \tag{1.42}$$

K independent of ε, but also of a and b. But by a linear change of variables (1.42) can be reduced to the case $a = -1$, $b = 1$, and for this standard case, the validity of (1.42) with a given K for a fixed interval is a special case of Lemma 6.

Finally, to obtain (1.38) from (1.42), let $a_k = (1 - 2^{-k})^{1/2}$ for every non-negative integer k, and write $\int_0^1 (1 - s^2)^{jp} |D_s v|^p ds$ as

$$\sum_{k=0}^{\infty} \int_{a_k}^{a_{k+1}} (1 - s^2)^{jp} |D_s v|^p ds\,.$$

We note that

$$\int_{a_k}^{a_{k+1}} (1 - s^2)^{jp} |D_x v|^p ds \leqq (1 - a_k^2)^{jp} \int_{a_k}^{a_{k+1}} |D_s v|^p ds \leqq$$

$$\leqq K \left\{\int_{a_k}^{a_{k+1}} (1 - a_k^2)^{2jp} |D_s^2 v|^p ds\right\}^{1/2} \left\{\int_{a_k}^{a_{k+1}} |v|^p ds\right\}^{1/2} \leqq$$

$$\leqq \varepsilon \int_{a_k}^{a_{k+1}} (1 - s^2)^{2jp} |D_s^2 v|^p ds + K\varepsilon^{-1} \frac{(1 - a_k^2)^{jp}}{(1 - a_{k+1}^2)} \int_{a_k}^{a_{k+1}} |v|^p ds\,.$$

But since the last coefficient is bounded uniformly in k, summing over k, we obtain (1.38) and the Lemma is proved.

Section 2: The Dirichlet Problem

Let $A = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha$ be a partial differential operator of order $2m$, $(m \geqq 1)$, on an open subset G of E^n which is uniformly regular of class C^{2m}, Γ the boundary of G. By Lemma 4 of Section 1, for $|\beta| < 2m$, there exists a bounded linear mapping $D^\beta|_\Gamma$ of $W^{2m,p}(G)$ into $W^{2m-1-|\beta|,p}(\Gamma)$ such that for u in $R(G)$, $D^\beta u|_\Gamma$ is the ordinary boundary-value function of $D^\beta u$.

Definition 3. *$D(A_p)$, for $1 < p < \infty$ and any partial differential operator A of order $2m$, the domain of the realization of A in $L^p(G)$ under null Dirichlet boundary conditions of order m on Γ, is given by*

$$D(A_p) = \{u : u \in W^{2m,p}(G),\ D^\beta u|_\Gamma = 0 \text{ for } |\beta| < m\}\,.$$

Definition 4. *The operator A_p with dense domain in $L^p(G)$ and range in $L^p(G)$, $1 < p < \infty$, has domain $D(A_p)$ as defined above, with $A_p u = Au$, for u in $D(A_p)$. (Au, of course, is well defined for u in $W^{2m,p}(G)$ and lies in $L^p(G)$.)*

It is the main purpose of the present paper to describe the spectrum and resolvent of the operator A_p in $L^p(G)$ when A is regularly elliptic in the following sense:

Definition 5. *A is said to be regularly elliptic on G if A is uniformly elliptic on G, and if, in addition, at each boundary point x_0 of G with N the normal vector to Γ at x_0 and t any unit tangent vector to Γ at x_0, the polynomial $a(x, t+\lambda N)$ in the complex variable λ has exactly m roots in the upper λ-half-plane.*

Remark. If $n>2$, every uniformly elliptic differential operator A is regularly elliptic. For general n, every uniformly elliptic A which is essentially real is regularly elliptic.

Theorem 1. *For $1<p<\infty$, let G be uniformly regular of class C^{2m}, A a regularly elliptic differential operator in G satisfying the regularity conditions* (I) *and* (II) *of Section* 1 *on its coefficients. Then A_p is a closed operator in $L^p(G)$.*

We shall derive Theorem 1 from the following more precise result, which in effect asserts that the closedness of A_p is uniform over all regularly elliptic operators A for which the constants of ellipticity and the data of (I) and (II) are uniform:

Theorem 2. *Let G be an open set in E^n uniformly regular of class C^{2m}, $A=\sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$ a regularly elliptic differential operator on G, with $|a(x,\xi)|\geq$ $\geq c_0|\xi|^{2m}$ for x in G and any real n-vector ξ. Suppose that all α, $|a_\alpha(x)|\leq M$ on G, while $|a_\alpha(x)-a_\alpha(y)|\leq w(\delta)$ for $|x-y|\leq\delta$, x, y in G, where the function $w(\delta)\to 0$ as $\delta\to 0$. Then there exists a constant $K_p>0$, depending only on p, G, c_0, M, and the function $w(\delta)$ such that for u in $D(A_p)$,*

$$\|u\|^p_{W^{2m,p}(G)}\leq K_p\{\|A_pu\|^p_{L^p(G)}+\|u\|^p_{L^p(G)}\}. \tag{2.1}$$

Proof of Theorem 1 using Theorem 2. Let G_p be the graph of A_p realized in $L^p(G)\times L^p(G)$, where the latter space is normed by $\|(u,v)\|=\{\|u\|^p_{0,p}+$ $+\|v\|^p_{0,p}\}^{1/p}$. The closedness of A_p is equivalent by definition to the fact that G_p is a closed subset of $L^p(G)\times L^p(G)$. Consider the mapping T of G_p into $W^{2m,p}(G)$ defined by $T(u, A_pu)=u$. By (2.1), T is bounded and hence may by extended by continuity to the closure of G_p in $L^p(G)\times L^p(G)$. The extended values of T lie in the closure of $D(A_p)$ in $W^{2m,p}(G)$, but since $D^\beta|_\Gamma$ is a continuous linear map of $W^{2m,p}(G)$ into $W^{2m-1-|\beta|,p(\Gamma)}$ it follows that $D(A_p)$ is closed in $W^{2m,p}(G)$ and the extended values of $T(u,v)$ lie in $D(A_p)$. For u in $D(A_p)$, however, $A_pT(u,A_pu)=A_pu$. Since A_p is a continuous mapping of $W^{2m,p}(G)$ into $L_p(G)$, it follows that for the extended values of T, $A_pT(u,v)=v$. Thus for (u,v) in $cl(G_p)$, $u\in D(A_p)$, $v=A_pu$, i.e. G_p is closed, and, thereby, A_p as well.

Theorem 2 is an a-priori estimate of a type which has been studied recently by a number of writers (as we remark below). In its precise form as given here particularly for an unbounded domain G, no proof in papers in print at the time of writing has appeared. (Cf., however, the writer's forthcoming paper [16], where similar a-priori estimates are obtained for the general class of regular boundary-value problems on uniformly regular domains.) We give below a simple derivation of Theorem 2 from the following particularly simple result

for an equation with constant coefficients on either of two standard bounded domains:

Lemma 8. *Let B_0 be the unit ball in E^n, $B_0 = \{y : |y| < 1\}$, $P_0 = B_0 \cap \{y : y_1 = 0\}$, $B_0^+ = B_0 \cap \{y : y_1 > 0\}$. Suppose A_0 is a homogeneous regularly elliptic differential operator with constant coefficients, $A_0 = \sum_{|\alpha| = 2m} a_{\alpha,0} D^\alpha$, with $\left| \sum_{|\alpha| = 2m} a_{\alpha,0} \xi^\alpha \right| \geqq c_0 |\xi|^{2m}$ for every real n-vector ξ and some $c_0 > 0$. Then for $1 < p < \infty$ there exists $K_p(c_0) > 0$ such that:*

(a) If u lies in $W^{2m,p}(B_0)$ and has compact support in B_0, then

$$\|u\|_{W^{2m,p}(B_0)} \leqq K_p(c_0) \, \|A_0 u\|_{L^p(B_0)} . \tag{2.2}$$

(b) If u lies in $W^{2m,p}(B_0^+)$, $D^\beta u|_{P_0} = 0$ for $|\beta| < m$, and if the support of u is a compact subset of $B_0^+ \cup P_0$, then

$$\|u\|_{W^{2m,p}(B_0^+)} \leqq K_p(c_0) \, \|A_0 u\|_{L^p(B_0^+)} . \tag{2.3}$$

For the case of a strongly elliptic operator A and $p = 2$, Lemma 8 follows from the more general estimates of GUSEVA [33] and BROWDER [6], (and partly from NIRENBERG [52]: a detailed exposition of the methods used in [6] and [52] is given by LIONS in [45]). For $p = 2$ and the Dirichlet problem for an elliptic operator A, the inequality (2.3) was obtained by SCHECHTER [58]. For general p, the inequality (2.2) is an easy consequence of the well-known CALDERON-ZYGMUND inequalities for singular integral operators ([20], [21]) together with the construction of a fundamental solution for a homogeneous elliptic equation with constant coefficients (JOHN [37]). For $m = 1$ and strongly elliptic A, (2.3) was obtained first for general p by KOSHELEV [40]. For $m > 1$, and general p, proofs of (2.3) have been announced by the writer ([7]) and KOSHELEV and a proof for strongly elliptic A was published by KOSHELEV in [41]. The writer's proof for a more general class of boundary-value problems will appear in [17] (see also [14], [16]). All the above papers establish variants of Theorem 2, under various hypotheses on G.

We give below a simple derivation of Theorem 2 from Lemma 8, and refer for the latter result to the various papers mentioned above.

Proof of Theorem 2. Suppose u lies in $D(A_p)$ for a given p with $0 < p < 1$. For a given $\delta > 0$, to be specified later, let $\{N_k\}$ be the covering of G by open subsets of E^n of diameter $< \delta$ which is described in Lemma 2, $\{\eta_k\}$ the corresponding family of functions in $C_c^{2m}(N_k)$, $\{\Phi_k\}$ the corresponding family of homeomorphisms mapping each N_k onto the unit ball B_0 in E^n.

We remark that it is sufficient to prove that

$$\|\eta_k^2 u\|^p_{W^{2m,p}(N_k \cap G)} \leqq K_p \{\|A(\eta_k^2 u)\|^p_{L^p(G)} + \|\eta_k^2 u\|^p_{L^p(G)}\} , \tag{2.4}$$

with a constant K_p independent of k. Indeed, if we sum (2.4) over k, we obtain

$$\begin{aligned} \|u\|^p_{W^{2m,p}(G)} &\leqq K \left\{ \sum_k \|\eta_k^2 u\|^p_{W^{2m,p}(G)} \right\} \leqq \\ &\leqq K_1 \left\{ \sum_k \|A(\eta_k^2 u)\|^p_{L^p(G)} + \sum_k \|\eta_k^2 u\|^p_{L^p(G)} \right\} \leqq \\ &\leqq K_2 \left\{ \sum_k \|\eta_k^2 A u\|^p_{L^p(G)} + \|u\|^p_{2m-1,p} + \|u\|^p_{0,p} \right\} , \end{aligned} \tag{2.5}$$

by applying Lemma 3 and Lemma 2(c). By Lemma 6,

$$\|u\|^p_{2m-1,p} \leqq (2K_2)^{-1}\|u\|^p_{2m,p} + K_3\|u\|^p_{0,p},$$

while

$$\sum_k \|\eta_k^2 A u\|^p_{0,p} \leqq \|A u\|^p_{0,p}.$$

Thus (2.5) yields

$$\|u\|^p_{2m,p} \leqq K_4\{\|A u\|^p_{0,p} + \|u\|^p_{0,p}\} + \tfrac{1}{2}\|u\|^p_{2m,p}, \tag{2.6}$$

from which (2.1) follows.

Replacing u by $\eta_k^2 u$, we may assume without loss of generality that u has compact support in N_k. Consider the function v on B_δ or B_δ^+, respectively, defined by $v(y) = u(\Psi_k(y))$, defined on $\Phi_k(N_k \cap G)$. By Lemma 2, $\|u\|_{2m,p} \leqq$ $\leqq K\|v\|_{2m,p}$, $\|v\|_{0,p} \leqq K\|u\|_{0,p}$, with a constant K independent of k or δ, while

$$(A u)(\Psi_k(y)) = A_y\{u(\Psi_k(y))\}, \tag{2.7}$$

where A_y is an elliptic differential operator on $\Phi_k(N_k \cap G)$ whose characteristic form $a(y, \xi) = a(x_k(y), \xi')$, where $\xi'_j = \sum_h c_{jh}(y)\,\xi_h$, and $c_{jh}(y)$ $= (i D_j \Phi_{kh})(\Psi_k(y))$, and with the coefficients of A_y uniformly bounded by a multiple c of the bound on the $|a_\alpha(x)|$, c independent of k or δ. Moreover, since the top order coefficients of A_y are obtained from those of A by a linear transformation with uniformly bounded coefficients, the oscillation of the coefficients of A_y on B_δ (or B_δ^+) is less than $c_1 \sum_\alpha \operatorname{osc}_{N_k} |a_\alpha|$, where c_1 is independent of both k and δ. Finally, $\|A_y v\|_{0,p} \leqq K\|A u\|_{0,p}$ so that it suffices, in order to establish (2.4), to prove that

$$\|v\|^p_{2m,p} \leqq K\{\|A_y v\|^p_{0,p} + \|v\|^p_{0,p}\}. \tag{2.8}$$

Note that v is defined on and has compact support in B_δ (or B_δ^+) and choosing δ sufficiently small for any given $\varepsilon > 0$, that $\sum_{|\alpha|=2m} |a_\alpha(y) - a_\alpha(y)| \leqq \varepsilon$, for y, and y in B_δ or B_δ^+, respectively where $A_y = \sum_{|\alpha| \leqq 2m} a_\alpha(y) D^\alpha$. In particular, we suppose that $\varepsilon < (2K_p)^{-1}$, where K_p is the constant of the inequalities in Lemma 7. In addition, in the case of B_δ^+, we assume that v satisfies the null Dirichlet boundary conditions on $P_\delta = B_\delta \cap \{y_1 = 0\}$.

We remark that because of the homogeneity of the differential operator A_0, Lemma 8 is valid on the domains B_δ and B_δ^+ with the same constants K_p as on the domains B_0 and B_0^+, respectively. Applying (2.2) and (2.3) to v and letting $A_0 = \sum_{|\alpha|=2m} a_\alpha(0) D^\alpha$, we obtain,

$$\|v\|_{2m,p} \leqq K_p\|A_0 v\|_{0,p} \leqq K_p\Big\{\|A_y v\|_{0,p} +$$

$$+ \Big\|\sum_{|\alpha|=2m} (a_\alpha(y) - A_\alpha(0)) D^\alpha v\Big\|_{0,p} + \Big\|\sum_{|\alpha|<2m} a_\alpha(y) D^\alpha v\Big\|_{0,p}\Big\} \leqq$$

$$\leqq K_p\{\|A_y v\|_{0,p} + (2K_p)^{-1}\|v\|_{2m,p} + C_1\|v\|_{2m-1,p}\}.$$

It follows from (2.9) that

$$\|v\|_{2m,p} \leqq K(c_0, M)\{\|A_y v\|_{2m,p} + \|v\|_{2m-1,p}\}. \tag{2.10}$$

By Lemma 6, however, $\|v\|_{2m-1,p} \leqq (2K(c_0, M))^{-1}\|v\|_{2m,p} + K_1\|v\|_{0,p}$, and substituting into (2.10), we obtain (2.8) and Theorem 2 is proved.

For our further discussion, we need the following sharpened form of Theorem 2:

Theorem 3. *Let G be an open set in E^n with boundary Γ, G uniformly regular of class C^{2m}. Let $A = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha$ be a regularly elliptic differential operator on G with $|a(x, \xi)| \geqq c_0|\xi|^{2m}$ for x in G, ξ a real n-vector. ($c_0 > 0$.) Suppose that the coefficients a_α are uniformly bounded on G, while a_α is uniformly continuous on G for $|\alpha| = 2m$. Suppose that u is a function in $L^p(G)$ having the property that for every point x_0 on $G \cup \Gamma$, there is a neighborhood N of x_0 such that $u \in W^{2m,p}(N)$. Suppose further that for $N \cap \Gamma \neq \emptyset$ u satisfies the null Dirichlet boundary conditions of order m on $N \cap \Gamma$, and that Au lies in $L^p(G)$.*

Then u lies in $W^{2m,p}(G)$ and inequality (2.1) holds for u.

Proof of Theorem 3. In Lemma 2, we established the existence of a covering of G by open sets $\{N_k\}$ of E^n having the following properties:

(1) Any $(R+1)$ distinct sets of $\{N_k\}$ have an empty intersection.

(2) Each N_k is the image under a homeomorphism Ψ_k of the ball B_δ, for some fixed $\delta > 0$, with inverse Φ_k. The mappings Φ_k and Ψ_k are of class C^{2m} and satisfy the conditions of Lemma 2(c).

(3) There exists a constant θ, with $0 < \theta < 1$, such that $G = \bigcup_k N_k'$, where $N_k' = \Psi_k\{y : |y| < \theta\delta\}$.

Let B_1 be the ball about the origin in E^n of radius $(n^{1/2} + 1)$, $B_0 = \{y : |y| < 1\}$. There exists a homeomorphism Ψ_0 of class C^{2m} of B_1 on B_δ carrying B_0 onto $\{y : |y| < \theta\delta\}$ with an inverse Φ_0 of class C^{2m}. Let $\zeta(y) = \prod_{j=1}^{n}(1 - y_j^2)$ be the function of Lemma 7. If $\zeta_k(x) = \zeta(\Phi_0(\Phi_k(x)))$ for x in N_k, $\zeta_k(x) = 0$ for x outside N_k, and the function ζ_k lies in $C_c^{2m}(E^n)$.

Consider the function u of our Theorem. Since u lies in $W^{2m,p}(N)$ for some neighborhood N (not necessarily uniform) in G of every point of $G \cup \Gamma$, it follows that u lies in $W^{2m,p}(N_k \cap G)$ for each of the neighborhoods N_k defined above, and indeed in $W^{2m,p}(F \cap G)$ where F is any bounded open set of E^n. Let us verify the inequalities,

$$\sum_{|\beta|=j} \|\zeta_k^j D^\beta u\|_{0,p}^p \leqq K\Big\{\sum_{|\beta| \leqq 2m} \|\zeta_k^{|\beta|} D^\beta u\|_{0,p}^p\Big\}^{j/2m} \|u\|_{L^p(N_k \cap G)}^{\frac{p(2m-j)}{2m}} \tag{2.11}$$

which corresponds to (1.35) of Lemma 7, with a constant K independent of k. The inequality (2.11) is equivalent by the argument of Lemma 6 to,

$$\sum_{|\beta|=j} \|\zeta_k^j D^\beta u\|_{0,p}^p \leqq \varepsilon \sum_{|\beta| \leqq 2m} \|\zeta_k^{|\beta|} D^\beta u\|_{0,p}^p + K_1\varepsilon^{-\mu}\|u\|_{L^p(N_k \cap G)}^p, \quad (\text{all } \varepsilon > 0), \tag{2.12}$$

with $\mu = j(2m-j)^{-1}$. But (2.12) follows from the corresponding variant of (1.35) by transplanting ζ_k and u to B_1 by the homomorphism $\Psi_0 \cdot \Psi_k$ and using the inequalities of Lemma 2(c).

On the other hand, $A(\zeta_k^{2m}u) = \zeta_k^{2m}Au + \sum_{|\beta|<2m} c_{k,\beta}(x)\zeta^{|\beta|}D^\beta u$, where the coefficients $c_{k,\beta}(x)$ are uniformly bounded independently of k. Since $u \in W^{2m,p}(N_k \cap G)$ and $\zeta_k^{2m} \in C_c^{2m}(N_k)$, $\zeta_k^{2m}u$ lies in $W^{2m,p}(G)$ and satisfies the null Dirichlet boundary conditions of order m on Γ. Applying Theorem 2 to $\zeta_k^{2m}u$, we obtain (taking all norms on $N_k \cap G$),

$$\begin{aligned} \|\zeta_k^{2m}u\|_{2m,p}^p &\leqq K\{\|A(\zeta_k^{2m}u)\|_{0,p}^p + \|\zeta_k^{2m}u\|_{0,p}^p\} \leqq \\ &\leqq K_2\{\|\zeta_k^{2m}Au\|_{0,p}^p + \sum_{|\beta|\leq 2m-1}\|\zeta_k^{|\beta|}D^\beta u\|_{0,p}^p\}. \end{aligned} \tag{2.13}$$

We observe, however, that

$$\begin{aligned} \|\zeta_k^{2m}u\|_{2m,p}^p &= \sum_{|\alpha|\leq 2m}\|D^\alpha(\zeta_k^{2m}u)\|_{0,p}^p \geqq \\ &\geqq \sum_{|\alpha|\leq 2m}\|\zeta_k^{|\alpha|}D^\alpha u\|_{0,p}^p - K_3\sum_{|\beta|\leq 2m-1}\|\zeta_k^{|\beta|}D^\beta u\|_{0,p}^p, \end{aligned} \tag{2.14}$$

$$\|\zeta_k^{2m}Au\|_{0,p}^p \leqq \|Au\|_{0,p}^p, \tag{2.15}$$

and combining (2.13), (2.4), and (2.15) with (2.12) taken for ε sufficiently small, we obtain finally

$$\sum_{0<|\alpha|\leq 2m}\|\zeta_K^{|\alpha|}D^\alpha u\|_{0,p}^p \leqq K_4\{\|Au\|_{L^p(N_k\cap G)}^p + \|u\|_{L^p(N_k\cap G)}^p\}. \tag{2.16}$$

But $\zeta_k(x) = 1$ on N_k', so that

$$\|u\|_{W^{2m,p}(N_k'\cap G)}^p \leqq K_4\{\|Au\|_{L^p(N_k\cap G)}^p + \|u\|_{L^p(N_k\cap G)}^p\}. \tag{2.17}$$

Summing over k, since u and Au lie in $LP(G)$, $\{N_k'\}$ is a covering of G and at most R of the sets N_k contain any given point x of G we obtain

$$\|u\|_{W^{2m,p}(G)} < \infty, \tag{2.18}$$

(2.1) holds and Theorem 3 has been established.

Another natural definition of the functions in $L^p(G)$ which satisfy the null Dirichlet condition of order m on Γ is the following

Definition 6. *The space $D_{p,m}$ of functions in $L^p(G)$ which satisfy the null Dirichlet boundary condition of order m on Γ in the variational sense is defined by*: $D_{p,m} = \{u : u \in W^{m,p}(G)$; *there exists a sequence $\{f_k\}$ in $C_c^\infty(G)$ such that $\|u - f_k\|_{W^{m,p}(G)} \to 0$ as $k \to \infty\}$.*

Lemma 9. *If G is an open set of E^n which is uniformly regular of class C^{2m}, then for $m \geqq 1$, $1 < p < \infty$, and any differential operator A of order $2m$ on G, $D(A_p) = D_{p,m} \cap W^{2m,p}(G)$.*

Proof of Lemma 9. The fact that $D_{p,m} \cap W^{2m,p}(G) \subset D(A_p)$ is an immediate consequence of Lemma 4 and the definition of $W^{2m,p}(G)$. Indeed, suppose $\{f_k\}$ is a sequence from $C_c^\infty(G)$ converging to u in $W^{m,p}(G)$ where u lies in $W^{2m,p}(G)$. By Lemma 4, since every set G regular of class C^{2m} is also regular of class C^j for $j < 2m$, $D^\beta f_k|_\Gamma$ converges to $D^\beta u|_\Gamma$ for $\beta < m$, where $D^\beta u|_\Gamma$ is understood in the extended sense as defined either on $W^{2m,p}(G)$ or

$W^{m,p}(G)$. (For $|\beta| < m$, the two definitions are obviously equal on $W^{2m,p}(G)$.) But $D^\beta f_k|_\Gamma = 0$ for all k so that $D^\beta u|_\Gamma = 0$ for $|\beta| < m$, and u lies in $D(A_p)$.

To prove the converse, suppose $u \in D(A_p)$. If we can approximate u in $W^{m,p}(G)$ by an element u_1 of $W^{2m,p}(G)$ with compact support in G, then since u_1 can be approximated arbitrarily closely in $W^{2m,p}(G)$ by its convolution with infinitely differentiable functions j_ε with small support, our result will be established. For fixed $\delta > 0$, let $\{N_k\}$ and $\{\Phi_k\}$ be two sequences of neighborhoods in E^n and homoemorphisms of N_k onto the disk B_δ, $\{\eta_k\}$ a sequence of functions with $\eta_k \in C_c^{2m}(N_k)$, with the three sequences satisfying the conditions of Lemma 2. If $u_k = \eta_k u$, u_k lies in $D(A_p)$ for every k. If we can find a function v_k in $W^{2m,p}(N_k \cap G)$ such that the support of v_k does not intersect $\Gamma \cap N_k$ and with $\|u_k - v_k\|_{W^{m,p}(N \cap G_k)} < 2^{-k}\varepsilon$, then for any large positive integer S, if we set $w_S = \sum_{k \leqq S} \eta_k v_k$, w_S must lie in $W^{2m,p}(G)$ and have compact support in G, while,

$$\|u - w_S\|_{W^{2m,p}} \leqq \left\| \sum_{k \leqq S} \eta_k (u_k - v_k) \right\|_{m,p} + \left\| \sum_{k > S} \eta_k^2 u \right\|_{m,p} \leqq K\varepsilon + e(S)\,, \tag{2.19}$$

where K is independent of S and ε, and $e(S) \to 0$ as $S \to \infty$. By choosing S sufficiently large, our conclusion would follow from (2.19). It thus suffices to consider the approximation of u_k by v_k, as above.

For $N_k \subset G$, u_k itself has compact support. We, therefore, need to consider only the case where $N_k \cap \Gamma \neq \Theta$, and by transplanting the function u_k by the homeomorphism Φ_k to the disk B_δ composed with the δ^{-1}-dilation, we may assume that u lies in $W^{2m,p}(B_0^+)$ and satisfies the Dirichlet null conditions $D^\beta u|_{P_0} = 0$ for $|\beta| < m$. Let us approximate u in $W^{m,p}(B_0^+)$ by a function v whose support does not intersect P_0. Let $\xi(y_1)$ be a function from $C_c^\infty(E^1)$ with $\xi(y_1) = 0$ for $y_1 \leqq 2^{-1}$, $\xi(y_1) = 1$ for $y_1 \geqq 1$, $0 \leqq \xi(y_1) \leqq 1$ for all y_1. For each positive integer s; let $\xi_s(y_1) = \xi(s\,y_1)$. If $v_s(y) = u(y) \cdot \xi_s(y_1)$, v_s lies in $W^{2m,p}(B_0^+)$, and the support of v_s is contained in the set $y_1 \geqq (2s)^{-1}$. We wish to show that $\|u - v_s\|_{W^{m,p}(B_0^+)} \to 0$ as $s \to \infty$. But for $|\beta| \leqq m$,

$$\begin{aligned} \|D^\beta(u - v_s)\|_{L^p(B_0^+)} &= \|D^\beta((1 - \xi_s)u)\|_{L^p(B_0^+)} \leqq \\ &\leqq \|D^\beta u\|_{L^p(B_0^+ \cap S_s)} + K \sum_{j_1 < j} s^{j - j_1} \sum_{|\alpha| = j_1} \|D^\alpha u\|_{L^p(B_0^+ \cap S_s)}\,, \end{aligned} \tag{2.20}$$

where $S_s = \{y : y_1 \leqq s^{-1}\}$, ($K$ independent of s).

To complete the proof, it suffices to show that

$$\|D^\alpha u\|_{L^p(B_0 \cap S_s)} = o(s^{m - |\alpha|}) \quad \text{as} \quad s \to \infty \quad \text{for} \quad |\alpha| \leqq m\,,$$

whenever $D^\beta u|_{P_0} = 0$ for all β with $|\beta| < m$. For $|\alpha| = m$, the result is obvious since u lies in $W^{m,p}(B_0^+)$. For the general case, it is a consequence of the following inequality:

$$\|D^\beta u\|^p_{L^p(B_0^+ \cap S_s)} \leqq K_p \{ s \|(D^\beta u|_{P_0})\|^p_{L^p(P_0)} + s^p \|(D_1 D^\beta u)\|^p_{L^p(B_0^+ \cap S_s)} \}\,, \tag{2.21}$$

valid for all u in $W^{m,p}(B_0^+)$. The inequality (2.21), which need be verified only for the dense subset $R(B_0^+)$, of $W^{m,p}(B_0^+)$, follows for smooth u from the

equation

$$(D^\beta u)(y_1, y) - (D^\beta u)(0, y) = \int_0^{y_1} i(D_1 D^\beta u)(t, y)\, dt\,,$$

by applying Hölder's inequality and integrating in y over $B_0^+ \cap S_s$.

In the remainder of this section, we consider modified Dirichlet problems defined with respect to a sub-division of the boundary into two pieces. Let Γ be the boundary of G, Γ_0 an open subset of Γ, $\Gamma_1 = \Gamma - \Gamma_0$. We denote by $W^{2m,p}_{\text{loc}}(G \cap \Gamma_0)$ the family of functions u in $L^p(G)$ such that u lies in $W^{2m,p}(N)$ for some neighborhood N in G of each point x_0 in $G \cup \Gamma_0$, and by $C_c^{2m}(G \cup \Gamma_0)$ the family of infinitely differentiable functions on the closure of G whose support in G is a compact subset of $G \cup \Gamma_0$. Let $1 < p < \infty$.

Definition 7. *With respect to the subdivision of Γ, $\Gamma = \Gamma_0 \cup \Gamma_1$, we define the operators $A_{p,0}$ and $A_{p,1}$ as follows:*

(a) $D(A_{p,0}) = \{u : u \in W^{2m,p}(G);\ D^\beta u|_{\Gamma_0} = 0$ *for* $|\beta| < m;\ D^\beta u|_{\Gamma_1} = 0$ *for* $|\beta| < 2m\}$,

$$A_{p,0} u = A u,\ u \in D(A_{p,0})\,.$$

(b) $D(A_{p,1}) = \{u : u \in W^{2m,p}_{\text{loc}}(G \cup \Gamma_0) : Au \in L^p(G);\ D^\beta u|_{\Gamma_0} = 0$ *for* $|\beta| < m\}$,

$$A_{p,1} u = A u,\ u \in D(A_{p,1})\,.$$

Theorem 1′. *If G is an open subset of E^n which is uniformly regular of class C^{2m}, Γ_0 an open subset of Γ, $\Gamma_1 = \Gamma - \Gamma_0$, A a regularly elliptic differential operator on G satisfying the regularity conditions* (I) *and* (II) *of Section* 1, *$1 < p < \infty$, then $A_{p,0}$ and $A_{p,1}$ are closed operators in $L^p(G)$.*

Proof of Theorem 1′. The proof is a minor variant of the proof of Theorem 1, and like the latter follows from a-priori estimates in $W^{2m,p}(G)$ and $W^{2m,p}_{\text{loc}}(G \cup \Gamma_0)$. Indeed, for $A_{p,0}$ the argument of Theorem 1 yields the proof directly. For $A_{p,1}$, we need the following:

Theorem 2′. *Under the hypotheses of Theorem* 2, *for every sub-domain C of G_0 having compact closure in $G \cup \Gamma_0$, there exists a constant $K(p, C)$ such that for u in $D(A_{p,1})$,*

$$\|u\|^p_{W^{2m,p}(C)} \leq K(p, C)\{\|A_{p,1} u\|^p_{L^p(G)} + \|u\|^p_{L^p(G)}\}\,. \tag{2.21}$$

The proof of Theorem 2′ follows with minor changes from the proof of Theorem 2 and will be left to the reader.

Lemma 10. *Let $\tilde{A}_{p,0}$ be the restriction of $A_{p,0}$ to $C_c^{2m}(G \cup \Gamma_0) \cap D(A_{p,0})$. Suppose that Γ_0 is regular in the following sense:*

If F_0 is the boundary of Γ_0 in Γ, there exists a regular covering $\{N_k\}$ of Γ in G of the type described in Definition 1 *such that for each k for which $N_k \cap F_0 \neq \Theta$, $\Phi_k(N_k \cap \Gamma_0)$ is either $\{y : |y| < 1,\ y_1 = 0,\ y_2 > 0\}$ or $\{y : |y| < 1,\ y_1 = 0\}$.*

Then $A_{p,0}$ is the closure in $L^p(G)$ of the linear map $\tilde{A}_{p,0}$.

Proof of Lemma 10. It suffices to show that $C_c^{2m}(G \cup \Gamma_0) \cap D(A_{p,0})$ is dense in $D(A_{p,0})$ in the space $W^{2m,p}(G)$. Using the functions $\lambda_k(x)$ of the proof of Lemma 1 and replacing u in $D(A_{p,0})$ by $\lambda_k u$, it suffices to assume that u has

bounded support. Since $u = \sum_k \eta_k^2 u$, where $\{\eta_k\}$ is the family of functions constructed in Lemma 2, replacing u by $\eta_k u$, we may assume that u has compact support in one of the sets N_k. Transplanting the function u by the mapping Φ_k, we need to consider only three typical situations

$$(B_0^+ = \{y : |y| < 1,\ y_1 > 0\},\ P_0 = \{y : |y| < 1,\ y_1 = 0\},$$

$$P_1 = \{y : |y| < 1,\ y_1 = 0,\ y_2 > 0\}.)$$

(1) $u \in W^{2m,p}(B_0^+)$; u has compact support in $B_0^+ \cup P_0$; $D^\beta u|_P = 0$ for $|\beta| < 2m$.

(2) $u \in W^{2m,p}(B_0^+)$; u has compact support in $B_0^+ \cup P_0$; $D^\beta u|_{P_1} = 0$ for $|\beta| < m$; $D^\beta u|_{P_0 - P_1} = 0$ for $|\beta| < 2m$.

(3) $u \in W^{2m,p}(B_0^+)$ with compact support in $B_0^+ \cup P_0$; $D^\beta u|_{P_0} = 0$ for $|\beta| < m$.

In case (1), we need to approximate u in $W^{2m,p}(B_0^+)$ by a function from $C_c^{2m}(B_0^+)$. In case (2), we need to approximate u in $W^{2m,p}(B_0^+)$ by a function v from $C_c^{2m}(B_0^+ \cup P_1)$ such that $D^\beta v|_{P_1} = 0$ for $|\beta| < m$, while in case (3), v needs only to lie in $R(B_0^+)$ and satisfy $D^\beta v|_P = 0$ for $|\beta| < m$.

The approximation in case (1) follows directly from the argument of the proof of Lemma 9, and we therefore consider only cases (2) and (3).

To treat case (2), we first remark that for sufficiently small positive h, $u_h(y) = u(y_2 - h, \hat{y}_2)$ is an arbitrarily good approximation to u in $W^{2m,p}(B_0^+)$, (where $\hat{y}_2 = (y_1, y_3, \ldots, y_n)$), u_h has compact support in $B_0^+ \cup P_0$, and satisfies the same boundary conditions as u on P_0 and P_1, indeed in a strengthened form, since $D^\beta u_h = 0$ for $|\beta| < 2m$ on P_0 whenever $y_2 > -h$. Replacing u by u_h for small h, we may therefore assume that u satisfies the null Cauchy boundary conditions of order $2m$ outside a compact subset of P_1. In case (3), u may be assumed null outside a compact subset of $P_0 \cup B_0^+$.

Let $j_h(x)$ be a sequence of functions in $C_c^\infty(E^{n-1})$ converging in the usual sense to the Dirac delta function, and form

$$u_h(x_1, \hat{x}) = \int_{E^{n-1}} j_h(\hat{y})\, u(x_1, \hat{x} + \hat{y})\, d\hat{y}\ .$$

It follows from a standard argument that $u_h \to u$ in $W^{2m,p}(B_0^+)$, that all the u_h for h sufficiently small have compact support in $P_0 \cup B_0^+$, and that the u_h satisfy the same boundary conditions on P_0 as does u. Replacing u by one of the u_h for small h and noting that each u_h and all its derivatives $D^\beta u_h|_{P_0}$ are infinitely differentiable functions on P_0, we may assume that the latter fact is true for u itself without any loss of generality. Let $u_j(x) = i^j D_1^j u|_{P_0}$, $h(x_1)$ a smooth function of x_1 which is 1 for x_1 near zero and 0 for $x_1 \geqq \varepsilon > 0$. If $v(x) = u(x) - \sum_{j=m}^{2m-1} h(x_1)\, u_j(x)\, x_1^j / j!$, it follows that v has compact support in $P_0 \cup B_0^+$ and satisfies the null Cauchy boundary conditions on P_0, i.e. $D^\beta v|_{P_0} = 0$ for $|\beta| < 2m$. From the argument of Lemma 9 it follows that v can be approximated in $W^{2m,p}(B_0^+)$ by a function w from $C_c^\infty(B_0^+)$. Thus u can therefore be

approximated by $w + \sum_{j=m}^{2m-1} h(x_1)\, x_1^j\, u_j(x)/j!$ which satisfies our given conditions, and our lemma is proved.

Theorem 4. *Let G be an open subset of E^n, A a regularly elliptic differential operator on G satisfying the hypotheses of Theorem 3. Suppose that in addition for all α and β with $|\alpha| \leqq 2m$, $|\beta| \leqq j$, $D^\beta a_\alpha$ is uniformly bounded on G, while G is uniformly regular of class C^{2m+j}, $(j \geqq 1)$. Then if $u \in D(A_p)$ and $Au \in W^{j,p}(G)$, it follows that u lies in $W^{2m+j,p}(G)$, and for each pair of open subsets B and C of G with $d(C, E^n - B) > 0$, there exists a constant C_j (independent of u) such that*

$$\|u\|_{W^{2m+j,p}(C)} \leqq C_j \{\|Au\|_{W^{j,p}(B)} + \|u\|_{L^p(B)}\}.$$

Proof of Theorem 4. The proof proceeds by induction.

Let $u \in D(A_p)$. Restricting u to one of the coordinate neighborhoods N_k of Lemma 2 and transplanting the restricted function to B_0 or B_0^+, we may assume that u lies in $W^{2m,p}(B_0)$ or $W^{2m,p}(B_0^+)$, and in the latter case satisfies the null Dirichlet boundary conditions $D^\beta u|_{P_0} = 0$ for $|\beta| < m$, while Au lies in $W^{j,p}(B_0)$ or $W^{j,p}(B_0^+)$, respectively. If C is a subset of B_0 or B_0^+ with compact closure in B_0 or B_0^+ P_0, respectively, B another such subset containing C in its interior in B_0 or $B_0^+ \cup P_0$, it suffices to show that $u \in W^{2m+j,p}(C)$ and that there exists an inequality of the form

$$\|u\|_{W^{2m+j,p}(C)} \leqq C_j \{\|Au\|_{W^{j,p}(G)} + \|u\|^p_{W^{2m-1+j,p}(B)}\}, \tag{2.22}$$

under the assumption that u lies in $W^{2m+j-1,p}(B)$, and the coefficients a_α having uniformly bounded derivatives of orders $\leqq j$. We suppose (2.22) has been established for all such operators A for $j < J$ $(J \geqq 1)$, and since (2.22) is certainly true for $j = 0$, it suffices to show that it is true for the given J.

We begin by replacing u by $v = \lambda(x)u$, where λ is a function from $C_c^\infty(E^n)$ which equals 1 on C and the restriction of whose support to B_0 or B_0^+ is contained in B. Since $Av = A(\lambda u) = \lambda Au + \sum_{|\beta| \leqq 2m-1} c_\beta(x) D^\beta u$, it follows from the assumption that c_β have bounded derivatives of order j on B, and u lies in $W^{2m+j-1,p}(B)$, that Av lies in $W^{j,p}(B)$ and its $W^{j,p}(B)$-norm is bounded by a constant multiple of

$$\{\|Au\|_{W^{j,p}(B)} + \|u\|_{W^{2m+j-1,p}(B)}\}.$$

Since v satisfies all the other conditions imposed on u, and since the problem of estimating the $W^{2m+j,p}(C)$-norm is the same for v as for u, we may replace u by v and assume that u itself has compact support in $B \cap P_0$. Set $f = Au$, and for $2 \leqq j \leqq n$, let $t_{j,h}$ by the translation mapping $u(x)$ into $u(x_j + h, \hat{x})$ $(\hat{x} = (x_1, \cdot, x_{j-1} x_{j+1} \ldots x_n)$ for any suitable small number h, $\Delta_{j,h} u = (t_{j,h} u - u) h^{-1}$ for $h \neq 0$. For h sufficiently small, $t_{j,h} u$ and $\Delta_{j,h} u$ both have compact support in $B \cup P_0$ and satisfy the null Dirichlet conditions on P_0 when P_0 lies on the boundary of B. Applying $\Delta_{j,h}$ to f, we obtain

$$\Delta_{j,h}(f) = \Delta_{j,h}(Au) = A(\Delta_{j,h} u) + \sum_{|\alpha| \leqq 2m} (\Delta_{j,h} a_\alpha)\, t_{j,h} D^\alpha u.$$

For each j and for $h \neq 0$, $\Delta_{j,h} u$ lies in $W^{2m+J-1}(B)$ and satisfies the null Dirichlet boundary conditions on P_0, if P_0 lies on the boundary of B. Applying the inductive hypothesis, we obtain

$$\|\Delta_{j,h} u\|_{W^{2m+J-1}(C)} \leqq C_{J-1}\{\|\Delta_{j,h}(Au)\|_{W^{J-1}} + \sum_{|\alpha| \leqq 2m} \|\Delta_{j,h} D^\alpha u\|_{W^{J-1,p}(C)}\} \tag{2.23}$$

since $D^\beta t_{j,h} a_\alpha$ is uniformly bounded for $|\beta| \leqq J-1$. In particular, it follows from (2.23) that $\Delta_{j,h} D^\beta u$ lie in a fixed bounded set of $L^p(C)$ for all j, $2 \leqq j \leqq n$, and all small $h \neq 0$. Since $\Delta_{j,h} D^\beta u$ converges to $D_j D^\beta u$ in the distribution sense as $h \to 0$ while a suitable subsequence converges weakly to an element of $L^p(C)$, it follows from the uniqueness of distribution limits that $D_j D^\beta u$ lies in $L^p(C)$, and by (2.23) satisfies the inequality

$$\|D_j D^\beta u\|_{L^p(C)} \leqq C'\{\|Au\|_{W^{J,p}(B)} + \|u\|_{W^{2m+J-1,p}(B)}\}. \tag{2.24}$$

Thus all the derivatives of u of order $2m+J$ are estimated by (2.24) except $D_1^{2m+J} u$. On the other hand, $Au = a_0 D_1^{2m} + R$, where R contains terms involving at least one tangential differentiation and a_0 lies in $C^J(B)$, is uniformly bounded away from zero on B, and has $D^\beta a_0$ uniformly bounded for $|\beta| \leqq J$. Applying D_1^J to Au, which lies in $W^{J,p}(B)$, we obtain $D_1^{2m+J} u = D_1^J(a_0^{-1} Au + a_0^{-1} Ru)$, with the right hand side already shown to be in $L^p(C)$. It follows that $D_1^{2m+J} u$ lies in $L^p(C)$, that (2.22) holds for $j = J$, and Theorem 4 is proved.

For use at a later point in our discussion, we give below one further result from the regularity theory of solutions of elliptic boundary value problems:

Lemma 11. *Let G be a domain in E^n whose boundary Γ is locally of class C^{4m}, A an elliptic differential operator on G with infinitely differentiable coefficients in G. Let Γ_0 be a part of the boundary of Γ satisfying the regularity conditions of Lemma 10. Suppose that u is an element of $W^{2m,2}_{\mathrm{loc}}(G \cup \Gamma_0)$, $f \in L^2_{\mathrm{loc}}(G \cup \Gamma_0)$, with*

$$(Av, Au) = (v, f) \tag{2.25}$$

for all v in $R(G) \cap D(A_{2,0})$. Then u lies in $W^{4m,2}_{\mathrm{loc}}(G \cup \Gamma_0)$.

Proof of Lemma 11. If we restrict our attention to a neighborhood of each point in G or on Γ_0, the smoothness condition on the coefficients implies that $\{\|Av\|^2 + \|v\|^2\}^{1/2}$, (norms taken in L^2), dominates a multiple of the $W^{2m,2}$-norm for all v in $D(A_{2,0})$. Using the fact that Γ is locally of class C^{4m}, we may transplant u restricted to suitable small neighborhoods to balls or half-balls in E^n, with parallel conditions holding for the transplanted functions. The conclusion of the Lemma follows from the estimation technique for $W^{J,2}$-norms of solutions of variational boundary problems as set forth in F. E. Browder [6] and L. Nirenberg [52]. (A detailed exposition of this method is given by J. L. Lions in [45].) On the sub-space $V = D(A_{p,0})$ of $W^{2m,2}$, we consider the coercive form $(Av, Av) + (v, v)$ on V, apply tangential differences to the function u (which has been multiplied by a suitable testing function to put it in $D(A_{2,0})$) to estimate its tangential derivatives, and estimate normal derivatives from the tangential derivatives together with the differential equation

$A'Au = f$ using Fourier transform methods. For the details of the argument, we refer to the above references. (We remark that the particular case considered here has the simplifying feature that the space V on the half-ball is invariant under local tangential translations.)

Lemma 12. *Let A be a uniformly elliptic partial differential operator of order $2m$ on an open set G of E^n with boundary Γ locally of class C^{2m}. Suppose that A has infinitely differentiable coefficients on the closure of G, and let A' be the formal adjoint operator of A. Let u be a function from $W^{2m,p}_{\mathrm{loc}}(G \cup \Gamma)$ for some p with $1 < p < \infty$, and $A^{\#}$ a differential operator with smooth coefficients such that*

$$(u, Af) = (A^{\#}u, f) \tag{2.26}$$

for all f in $R(G) \cap D(A_2)$.

Then $D^{\beta}u|_{\Gamma} = 0$ for $|\beta| < m$, and $A^{\#}u = A'u$.

Proof of Lemma 12. The fact that $A^{\#}u = A'u$ follows immediately from (2.26), since for $f \in C_c^{2m}(G)$, we also have

$$(u, Af) = (A'u, f)\,, \tag{2.27}$$

and subtracting (2.27) from (2.26), we obtain $((A^{\#} - A')u, f) = 0$ for all testing functions f, and hence $A^{\#}u = A'u$.

Let x_0 be a point on the boundary Γ of G, Φ a homeomorphism of class C^{2m} of a neighborhood N of x_0 on the unit ball B_0 in E^n such that $\Phi(N \cap G) = B_0^+$, Ψ the inverse mapping of B_0 onto N. Suppose $f \in R(N) \cap D(A_2)$. If $g(y) = f(\Psi(y))$, $(Af)(x(y)) = A_y g(y)$, where A_y is a uniformly elliptic differential operator on B_0^+, g is defined on B_0^+ and has null Dirichlet data on P_0. Conversely every such g with compact support in $B_0^+ \cup P_0$ may be so obtained. If $v(y) = u(\Psi(y))$, it follows from (2.26) that v satisfies the condition

$$(Jv, A_y g)_{B_0^+} = (Sv, g)_{B_0^+}\,, \tag{2.28}$$

where J is the Jacobian of the mapping Ψ, S is a differential operator on B_0^+ of order $2m$ with smooth coefficients, and (2.28) is valid for all g in $C_c^{2m}(B_0^+ \cup P_0)$ for which $D^{\beta}g|_{P_0} = 0$, $|\beta| < m$. It suffices to show that $v_1 = Jv$ satisfies the null Dirichlet conditions of order m on P_0.

By our preceeding remark, $Sv = A'_y v_1$, and (2.28) may be written as

$$(v_1, Ag) = (A'v_1, g)\,, \quad g \in C_c^{2m}(B_0^+ \cap P_0) \cap D(A_2)\,, \tag{2.29}$$

where we have omitted the sub-script in A_y without danger of confusion. It suffices to show that $D_1^j v_1|_{P_0} = 0$ for $0 \leqq j \leqq m-1$, and this latter fact is established by recursion on j, starting with $j = 0$ and continuing with increasing j. For smooth v_1 in $R(B_0^+)$ and hence for all v_1 in $W^{2m,2}(B_0^+)$ by continuity, we have

$$(v_1, Ag)_{B_0^+} = (A'v, g)_{B_0^+} + \sum_{\substack{|\alpha|+|\beta| \leqq 2m-1 \\ |\beta| \geqq m}} (c_{\alpha\beta}(x) D^{\alpha}v_1, D^{\beta}g)_{P_0} \tag{2.30}$$

for g as above with smooth coefficients $c_{\alpha\beta}$ which we prescribe in more detail below. Suppose that we have already established that $D_1^j v_1|_{P_0} = 0$ for all j with $0 < j < J$ where J is any fixed integer with $0 \leqq J \leqq m-1$. Let $g(y) = f(\hat{y}) \cdot y_1^{2m-1-J} h(y_1)$, where h is a smooth function which equals 1 for y_1 near zero, $h(y_1) = 0$ for $y_1 \geqq \varepsilon > 0$, $f \in C_c^\infty(E^{n-1})$. For the given g, $D_1^j g|_{P_0} = 0$ for $j = J$, while $D_1^J g|_{P_0} = c_J f(\hat{y})$, c_J a non-zero constant depending on m and J. It follows immediately by integration by parts from the inductive assumption that for any α different from $(2m, 0, \ldots, 0)$, $(v_1 a_\alpha D^\alpha g) - (D^\alpha(\bar{a}_\alpha v_1), g) = 0$. On the other hand $(v_1, a_0 D_1^{2m} g) - (D_1^{2m}(a_0 v_1), g) = c'_J(D_1^J v_1, a_0 f(\hat{y}))_{P_0}$, $c'_J \neq 0$. Therefore by (2.29), $(D_1^J v_1, a_0 f)_{P_0} = 0$ for every f in $C_c^\infty(E^{n-1})$, and since $a_0(x)$ is uniformly bounded away from zero, the set $\{a_0 f\}$ is dense in $L^2(P_0)$ and $D_1^J v_1|_{P_0} = 0$, completing the J-th step of the proof. The same argument is valid for $J = 0$, and the Lemma is proved.

Section 3: Adjoints

The principal topic of the present section is the study of the adjoint operator of A_p for $1 < p < \infty$, as defined in Section 2. We preface the discussion of the particular operators to be studied in detail by a brief summary of the definition and main properties of the adjoint of an unbounded linear operator in a Banach space. For a more detailed discussion of the latter, we refer to HILLE-PHILLIPS [34] or to a recent paper of the writer [13].

Let X be a complex Banach space, X^* its adjoint space, with the pairing between an element u of X and a linear functional u^* of X^* written as $\langle u, u^* \rangle$. Let T be a linear transformation (not necessarily continuous) with dense domain $D(T)$ in X and range $R(T)$ in X. The adjoint operator T^* to T is defined with domain $D(T^*) = \{u^* : u^* \in X^*, \langle Tu, u^* \rangle$ is a bounded linear functional on the dense subset $D(T)$ of $X\}$. For u^* in $D(T^*)$, there exists an unique element v^* of X^* such that $\langle u, v \rangle = \langle Tu, u^* \rangle$ for all u in $D(T)$, and we set $T^* u^* = v^*$. If T is a closed (or closable) operator, $D(T^*)$ will be weak*-dense in X^*. If in addition, X is reflexive, $D(T^*)$ will be dense for any closable operator T, and $(T^*)^*$ is the closure of T. For T closed, in particular, $(T^*)^* = T$. If $N(T)$ and $N(T^*)$ are the null-spaces of the closed operator T and its adjoint T^*, respectively, then $N(T) = \{u : u \in X, \langle u, T^* u^* \rangle = 0$ for all u^* in $D(T^*)\}$, $N(T^*) = \{u^* : u^* \in X^*, \langle Tu, u^* \rangle = 0$ for all u in $D(T)\}$. If $R(T)$, the range of T, is closed in X, then $R(T) = \{u : u \in X, \langle u, u^* \rangle = 0$ for all u^* in $N(T^*)\}$, $R(T^*)$ is closed in X^*, and $R(T^*) = \{u^* : \langle u, u^* \rangle = 0$ for all u in $N(T)\}$. (The equivalence of the closedness of $R(T)$ and $R(T^*)$ without reflexivity assumptions on X is proved in [13]).

The particular Banach spaces $L^p(G)$ which we consider below for $1 < p < \infty$ are all reflexive, and $(L^p(g))^* = L^q(G)$, where $q = p(p-1)^{-1}$. All the conclusions of the preceeding paragraph apply therefore to any closed linear operator T defined in these spaces, and in particular to the operators A_p, $A_{p,0}$ and $A_{p,1}$ defined in Section 2 for a regularly elliptic operator A of order $2m$ on an open subset G of E^n which is uniformly regular of class C^{2m} with respect to a subdivision of the boundary Γ of G into two pieces Γ_0 and Γ_1. We

shall suppose throughout that Γ_0 satisfies the regularity conditions of Lemma 10, and that A satisfies the regularity conditions (I) and (II) of Section 1.

A slightly sharper regularity assumption on the coefficients of A that we shall need in our discussion of adjoints is the following:

(IV) *There exists an open set G_0 of E^n containing G with $d(G, E^n - G_0) > 0$ such that A is defined and regularly elliptic on G_0, the coefficients a_α of A and a'_α of A' are uniformly bounded on G_0 for all α, and for $|\alpha| = 2m$, each a_α is uniformly continuous on G_0.*

The principal consequence of (IV) that we apply in the discussion below is the following:

Lemma 13. *Let A be a differential operator of order $2m$ on G satisfying* (IV), *A' its formal adjoint. Then there exists a sequence $\{A_k\}$ of differential operators with analytic coefficients on E^n such that if $A_k = \sum_{|\alpha| \leq 2m} a_{\alpha,k} D^\alpha$, A'_k is the formal adjoint of A_k, and $A'_k = \sum_{|\alpha| \leq 2m} a'_{\alpha,k}(x) D^\alpha$, then,*

(i) *For $|\alpha| = 2m$, $a_{\alpha,k}$ converges uniformly to a_α on G.*

(ii) *For all α, $|a_{\alpha,k}(x)| + |a'_{\alpha,k}(x)| \leq M$, independently of k, on G, while $a_{\alpha,k}(x) \to a_\alpha(x)$, $a'_{\alpha,k}(x) \to a'_\alpha(x)$ a.e. in G as $k \to \infty$.*

(iii) *The A_k are uniformly elliptic on E^n with a constant of ellipticity independent of k.*

(iv) *If A is uniformly strongly elliptic on G_0, A_k is uniformly strongly elliptic on E^n with constant independent of k. If A is essentially real, then all the A_k may be chosen essentially real.*

Proof of Lemma 13. We first approximate by a sequence $\{A_k\}$ with infinitely differentiable coefficients. The latter is obtained by choosing a sequence $\{j_k(x)\}$ from $C_c^\infty(E^n)$, as in Section 1, which converges to the Dirac delta function in the usual sense, and letting $a_{\alpha,k}(x) = \int_{G_0} j_k(x-y)\, a_\alpha(y)\, dy$. For the diameter of the support of $j_k < d(G, E^n - G_0)$, we have $D^\beta a_{\alpha,k}(x) = \int_{G_0} j_k(x-y)(D^\beta a_\alpha(y))\, dy$ for x in G. It follows that $a'_{\alpha,k}(x) = \int_{G_0} j_k(x-y) a'_\alpha(y)\, dy$ for x in G. The conclusions (i), (ii), (iii), and (iv) of the Lemma follow immediately from these integral representations by standard arguments.

For any fixed k, it follows from the representation $D^\beta a_{\alpha,k}(x) = \int_{G_0} (D^\beta j_k)(x-y)\, a_\alpha(y)\, dy$, that all the derivatives of $a_{\alpha,k}(x)$ are bounded on E^n. To complete the proof of the Lemma, it suffices to approximate the coefficients $a_{\alpha,k}(x)$ with arbitrary precision in the uniform $C^{2m}(E^n)$-norm by analytic functions. The latter may be accomplished, for example, by taking the convolution of $a_{\alpha,k}(x)$ with $\varepsilon^{-n}\pi^{-n/2}e^{-\varepsilon^2 r^2}$, for small $\varepsilon > 0$.

Lemma 14. *Let G be an open subset of E^n which is uniformly regular of class C^{2m}, A a differential operator of order $2m$ on G satisfying* (IV). *Let Γ_0 be a subset of the boundary Γ of G satisfying the regularity condition of Lemma* 10, *$A_{p,0}$ and $A_{p,1}$ the realizations of A in $L^p(G)$ satisfying null-Dirichlet conditions on Γ_0 and either null Cauchy boundary conditions or no boundary conditions on*

$\Gamma_1 = \Gamma - \Gamma_0$, $A'_{q,0}$ and $A'_{q,1}$ the corresponding realizations of the formal adjoint A' of A in $L^q(G)$ with $q = p(p-1)^{-1}$. Then we have:

(i) $A'_{q,1} \subseteq (A_{p,0})^*$, *i.e.* $(Af, g) = (f, A'g)$ *for* $f \in D(A_{p,0})$, $g \in D(A'_{q,1})$.

(ii) $A'_{q,0} \subseteq (A_{p,1})^*$.

As a particular case, when $\Gamma = \Gamma_0$,

(iii) $A'_q \subseteq (A_p)^*$.

Proof of Lemma 14. We remark first that if A satisfies (IV), so does A'. If (i) is established for A' and the exponent q, however, i.e. if $A_{p,1} \subseteq (A'_{q,0})^*$, then taking adjoints of both sides (and thereby reversing the contains relation), we have $A'_{q,0} = (A'_{q,0})^{**} \subseteq (A_{p,1})^*$, and (ii) follows. Since (iii) is a special case of (i), it suffices to prove (i) for all operators A satisfying the conditions of the Lemma and all p with $1 < p < \infty$.

We remark that if $f \in D(A_{p,0}) \cap R(G)$ with the support of f a compact subset of $G \cup \Gamma_0$ and if $g \in D(A'_{q,1}) \cap R(G)$, and, in addition, if the coefficients of A are infinitely differentiable on all of E^n, then the equality $(Af, g) = (f, A'g)$ follows from Green's formula (or more precisely, integration by parts). By Lemma 10 of Section 2, for each f in $D(A_{p,0})$ there exists a sequence $\{f_k\}$ from $D(A_{p,0}) \cap R(G)$ having compact support in $G \cup \Gamma_0$ with $f_k \to f$ in $W^{2m,p}(G)$. If for each k, we knew that $(Af_k, g) = (f_k, A'g)$ for the given g in $D(A'_{q,1})$, then taking the limit on k, we would obtain $(Af, g) = (f, A'g)$. It thus suffices to assume that f has compact support in $G \cup \Gamma_0$ and lies in $D(A_{p,0}) \cap R(G)$. Let h be a function from $C_c^\infty(E^n)$ such that h equals 1 on the support of f while the intersection of the support of h with G is a compact subset of $G \cap \Gamma_0$. Then $(Af, g) = (Af, hg)$, and $(f, A'g) = (f, A'(hg))$. Since hg lies in $D(A'_{q,0})$ if g lies in $D(A'_{q,1})$, we may assume that g itself lies in $D(A'_{q,0})$ and by the same argument as for f, that g lies in $D(A'_{q,0}) \cap R(G)$.

Finally, let $\{A_k\}$ be the sequence of approximating differential operators with infinitely differentiable coefficients constructed in Lemma 13. For each f in $D(A_{p,0})$, it follows from (ii) of Lemma 13 and dominated convergence, that $A_k f$ converges strongly to Af in $L^p(G)$. Similarly for g in $D(A'_{q,0})$, $A'_k g$ converges strongly to $A'g$ in $L^q(G)$. For each k, we have, by a preceeding remark, $(A_k f, g) = (f, A'_k g)$. Taking the limit on the inner product, we obtain $(Af, g) = (f, A'g)$, and the proof of the Lemma is complete.

When $p = 2$ (and thereby $q = 2$), we may immediately strengthen the conclusion of Lemma 14 to the following:

Theorem 5. *Let A be a regularly elliptic differential operator of order $2m$ defined on an open subset G of E^n, with G uniformly regular of class C^{2m} and locally of class C^{4m}. Suppose that A satisfies the regularity condition* (IV). *Let Γ_0 be a subset of the boundary Γ of G satisfying the regularity condition of Lemma* 10, *$A_{2,0}$ and $A_{2,1}$ the realizations of A in $L^2(G)$ with null Dirichlet conditions on Γ_0 and either null Cauchy conditions or no boundary conditions on $\Gamma_1 = \Gamma - \Gamma_0$. Let $A'_{2,0}$ and $A'_{2,1}$ be the corresponding realizations of the adjoint differential operator A'. Then:*

(i) $(A_{2,0})^* = A'_{2,1}$,

(ii) $(A_{2,1})^* = A'_{2,0}$.

As a particular case when $\Gamma = \Gamma_0$,

(iii) $(A_2)^* = A_2'$.

Proof of Theorem 5. We remark as in the proof of Lemma 14 that since A' satisfies the same conditions as A itself, (ii) may be obtained by taking the adjoint of (i), if the latter is taken for the operator A'.

Since (iii) is a special case of (i), it suffices to prove (i).

Next, we note that it suffices to prove that the mappings $T_1 = (A_{2,1}' A_{2,0} + I)$ and $T_2 = (A_{2,0}' A_{2,1}' + I)$ are onto all of $L^2(G)$. Indeed, by Lemma 14, $(A_{2,1}' A_{2,0} + I) \subseteq (A_{2,0}^* A_{2,0} + I)$, and the latter by a well-known theorem of von Neumann is an invertible mapping onto $L^2(G)$. If $(A_{2,1}' A_{2,0} + I)$ itself were onto, it would then follow that $(A_{2,1}' A_{2,0} + I) = (A_{2,0}^* A_{2,0} + I)$, i.e. $A_{2,0}' A_{2,0} = A_{2,0}^* A_{2,0}$. It would follow that $D(A_{2,0}^*) \cap R(A_{2,0}) = D(A_{2,1}) \cap R(A_{2,0}')$, while $A_{2,1}'$ and $(A_{2,0})^*$ are equal on this part of their domains. But if $v \in D(A_{2,0}^*)$, let u be the element of $D(A_{2,0} A_{2,1}')$ such that $A_{2,0} A_{2,1}' u + u = v$. Since u lies in $D(A_{2,1}')$ and $v \in D(A_{2,0}^*)$, it would follow that $A_{2,0} A_{2,1}' u = v - u$ lies in $D(A_{2,0}^*) \cap R(A_{2,0})$ and hence in $D(A_{2,1}')$. But then v, being the sum of two elements in $D(A_{2,1}')$, would itself lie in $D(A_{2,1}')$, i.e. $D(A_{2,0}^*) = D(A_{2,1}')$ and $A_{2,1}^* = A_{2,1}'$. Hence it suffices to prove that T_1 and T_2 above map onto $L^2(G)$.

Our next remark is that it suffices to prove the result for operators A with infinitely differentiable coefficients. Indeed, for a given operator A as in Theorem 5, consider the approximating sequence of operators $\{A_k\}$ with infinitely differentiable coefficients constructed in Lemma 13. Let $A_{2,0,k}$, $A_{2,1,k}$, $A_{2,0,k}'$, and $A_{2,1,k}'$ be the corresponding realizations of the operators A_k and A_k'. Suppose that for each k, the mappings $T_{1,k} = (A_{2,1,k}' A_{2,0,k} + I)$ and $T_{2,k} = (A_{2,0,k} A_{2,1,k}' + I)$ are onto all of $L^2(G)$. Let w be an element of $L^2(G)$, u_k and v_k the unique elements of $D(A_{2,1,k}' A_{2,0,k})$ and $D(A_{2,0,k} A_{2,1,k}')$ such that $T_{1,k} u_k = w$, $T_{2,k} v_k = w$. By Lemma 13, the operators A_k and A_k' satisfy conditions (I) and (II) and are regularly elliptic with all regularity constants independent of k. Let $u_k' = A_{2,0,k} u_k$, $v_k' = A_{2,1,k}' v_k$. We have $u_k' \in D(A_{2,1,k}')$, $v_k \in D(A_{2,0,k})$, and

$$A_{2,1,k}' u_k' + u_k = w \,. \tag{3.1}$$

$$A_{2,0,k} v_k' + v_k = w \,. \tag{3.2}$$

Taking the inner product of (3.1) with u_k, since $A_{2,1,k}' \subseteq A_{2,0,k}^*$, we have

$$(u_k', u_k') + (u_k, u_k) = (w, u_k) \,. \tag{3.3}$$

Similarly taking the inner product of (3.2) with v_k, we obtain

$$(v_k', v_k') + (v_k, v_k) = (v_k, w) \,. \tag{3.4}$$

Since obviously $\|(T_{1,k})^{-1}\| \leqq 1$, $\|(T_{2,k})^{-1}\| \leqq 1$, we have $\|u_k\| \leqq \|w\|$, $\|v_k\| \leqq \|w\|$. It follows from (3.3) and (3.4) that $\|u_k'\| \leqq \|w\|$, $\|v_k'\| \leqq w$.

Since $A_{2,0,k} u_k = u_k'$, it follows from Theorem 2 and the uniform regularity of the operators A_k that there exists a constant $K > 0$ and independent of k such that

$$\|u_k\|_{W^{2m,2}(G)} \leqq K\{\|u_k'\|_{0,2} + \|u_k\|_{0,2}\} \leqq 2K\|w\| \,. \tag{3.5}$$

Similarly by Theorem 2′ for every compact subset C of $G \cup \Gamma_0$, there exists a constant $K(C)$ such that

$$\|v_k\|_{W^{2m,2}(C)} \leqq 2K(C)\,\|w\|\,. \tag{3.6}$$

On the other hand, (3.1) implies by the above argument that

$$\|u_k'\|_{W^{2m,2}(C)} \leqq K(C)\,\|w\|\,, \tag{3.7}$$

while (3.2) implies that

$$\|v_k'\|_{W^{2m,2}(G)} \leqq K\|w\|\,. \tag{3.8}$$

Taking subsequences and applying the diagonal process, we obtain from the weak compactness of the unit ball in a Hilbert space that there exists a subsequence of the u_k, v_k, u_k', and v_k', which we may assume to be the original sequence such that u_k and v_k' converge weakly in $W^{2m,2}(G)$ while v_k and u_k' converge weakly in $W^{2m,2}(C)$ for every compact subset C of $G \cup \Gamma_0$. If we denote the limits of these respective sequences by u, v', v, and u' respectively, it follows that u and v' lie in $W^{2m,2}(G)$ while v and u' lie in $W^{2m,2}_{\text{loc}}(G \cup \Gamma_0)$. Since for every k, $u_k \in D(A_{2,0,k})$ and $v_k' D(A_{2,0,k}) = D(A_{2,0})$, while $D(A_{2,0})$ is a closed linear subspace of $W^{2m,2}(G)$ and hence weakly closed, it follows that u and v' lie in $D(A_{2,0})$. In addition, since the mapping from $W^{2m,2}(G)$ into the L^2-space of the Dirichlet data is strongly and hence weakly continuous, it follows that v and u' satisfy the null Dirichlet boundary condition of Γ_0.

Let G' be a bounded subdomain of G. By Lemma 13, the top order coefficients of A_k (and thereby A_k') converge uniformly to the corresponding top-order coefficients of A (and A'). The lower order coefficients $a_{\alpha,k}$ and $a'_{\alpha k}$ converge a.e. boundedly to a_α and a'_α. Since the sequence u_k converges weakly to u in $W^{2m,2}(G)$ and since by Lemma 5(b) of Section 1, the injection mapping of $W^{2m,2}(G)$ into $W^{2m-1,2}(G')$ for G' bounded is a compact linear mapping, it follows that u_k converges strongly to u in $W^{2m-1,2}(G')$. Consider $A_k u_k = \sum_{|\alpha|=2m} a_{\alpha,k} D^\alpha u_k + \sum_{|\alpha|<2m} a_{\alpha,k} D^\alpha u_k$. In the first sum, the uniform convergence of the coefficients $a_{\alpha,k}$ to a_α on G together with the weak convergence of $D^\alpha u_k$ to $D^\alpha u$ imply the weak convergence of the first sum to

$$\sum_{|\alpha|=2m} a_\alpha(x)\, D^\alpha u \quad \text{in} \quad L^2(G)\,.$$

For the second sum, the bounded almost everywhere convergence of $a_{\alpha,k}(x)$ to $a_\alpha(x)$ together with the strong convergence of $D^\alpha u_k$ to $D^\alpha u$ in $L^2(G')$ when $|\alpha| < 2m$ imply that the second sum converges strongly in $L^2(G')$ to $\sum_{|\alpha|<2m} a_\alpha(x)\, D^\alpha u$. Taken together, the total sum for $A_k u_k$ converges weakly in every $L^2(G')$ for G' bounded to Au. But $A_k u_k = u_k'$, and the latter converges in $L^2_{\text{loc}}(G \cup \Gamma_0)$ to u'. By the uniqueness of weak limits, it follows that $Au = u'$, and since u lies in $D(A_{2,0})$, we have $A_{2,0}u = u'$. Similarly $A_k v_k'$ converges weakly in $L^2(G')$ for bounded G' to Av', while by (3.2), $A_k v_k' = w - v_k$ converges in $L^2_{\text{loc}}(G \cup \Gamma_0)$ to $w - v$. By the uniqueness of weak limits once more, $Av' = w - v$, and since v' by our previous argument lies in $D(A_{2,0})$, $A_{2,0}v' = w - v$.

The same arguments apply to the sequences $\{v_k\}$ and $\{u'_k\}$, except that we must take weak limits on compact subdomains of G. Applying this argument together with (3.1), we obtain $A'v = v'$, and $A'u' = w - u$. Since v and u' both lie in $W^{2m,2}_{\text{loc}}(G \cup \Gamma_0)$, both satisfy the null Dirichlet boundary conditions of order m on Γ_0, and since $A'v$ and $A'u'$ lie in $L^2(G)$, it follows that v and u' lie in $D(A'_{2,1})$. Further, $A'_{2,1}v = v'$, $A'_{2,1}u' = w - u$.

We see finally that $u \in D(A_{2,0})$ and that $A_{2,0}u$ lies in $D(A'_{2,1})$, i.e. $u\, D(A'_{2,1}A_{2,0})$, and that $A'_{2,1}A_{2,0}u + u = w$. Similarly, v lies in $D(A'_{2,1})$, $A'_{2,1}v$ lies in $D(A_{2,0})$, i.e. v lies in $D(A_{2,0}A'_{2,1})$, and $A_{2,0}A'_{2,1}v + v = w$. In other terms, we have verified that T_1 and T_2 are onto mappings if the same fact is true for the differential operators A_k with infinitely differentiable coefficients.

Suppose then that A has infinitely differentiable coefficients on E^n. We wish to show that T_1 and T_2 are onto. For the first, let w be an arbitrary element of $L^2(G)$, and let u be the unique solution of the equation $(A^*_{2,0}A_{2,0} + I)u = w$. The function u obviously lies in $D(A_{2,0})$, and from the last equation it follows that

$$(A_{2,0}u, A_{2,0}v) + (u, v) = (w, v) \tag{3.9}$$

for any function v from $D(A_{2,0})$. It follows from Lemma 11 of Section 2 that u lies in $W^{4m,2}_{\text{loc}}(G \cup \Gamma_0)$, and hence that $A_{2,0}u$ lies in $W^{2m,2}_{\text{loc}}(G \cup \Gamma_0)$. If $u' = A_{2,0}u$, it follows from (3.9) that u' which lies in $W^{2m,2}_{\text{loc}}(G \cup \Gamma_0)$ satisfies the equation

$$(u', A_{2,0}v) = (w - u, v) \tag{3.10}$$

for all v in $D(A_{2,0})$, where $w - u$ lies in $L^2(G)$. Applying Lemma 12 of Section 2, it follows that u' satisfies the null Dirichlet conditions of order m on Γ_0 and $A'u' = w - u$ in G, i.e. $u' \in D(A'_{2,1})$, and $A'_{2,1}u' = w - u$. Putting the latter assertion in terms of u, u lies in $D(A'_{2,1}A_{2,0})$ and $(A'_{2,1}A_{2,0} + I)u = w$. We have thus proved that T_1 is onto.

The proof for T_2 runs along similar lines. Let v be the unique solution of the equation $(A'^*_{2,1})\, A'_{2,1}v + v = w$. The function v lies in $D(A'_{2,1})$, and taking the inner product of the latter equation with an arbitrary element f from $D(A'_{2,1})$, we obtain

$$(A'_{2,1}v, A'_{2,1}f) + (v, f) = (w, f)\,. \tag{3.11}$$

If we consider only f from the smaller class $D(A'_2)$ and apply Lemma 11 with A replaced by the operator A' and $\Gamma_0 = \Gamma$, we find that $v \in W^{4m,2}_{\text{loc}}(G \cup \Gamma)$. Setting $v' = A'_{2,1}v$, we see that $v' \in W^{2m,2}_{\text{loc}}(G \cup \Gamma)$ and satisfies the equation

$$(v', A'_{2,1}f) = (w - v, f) \tag{3.12}$$

for all $D(A'_{2,1})$. It follows once more from Lemma 12 that v' must satisfy the null Dirichlet boundary conditions of order m on Γ and hence $v' \in D(A_2)$. Let G'' be an open subset of E^n which contains G and whose boundary intersects the boundary of G in Γ_0, and extend v' and $w - v$ to G'' by setting them equal to zero outside of G. Since every g in $D(A'_{2,1})_{G''}$, where the latter operator is

considered on $L^2(G'')$, yields a function in $D(A'_{2,1})_G$ by restriction to G, it follows from (3.12) that

$$(v', A'_{2,1}g)_{G''} = (w - v, g)_{G''} \tag{3.13}$$

for all g in $W^{2m,2}(G'')$ satisfying the null Dirichlet conditions of order m on Γ_0. But A' has infinitely differentiable coefficients on G'', so that from the standard theorems for interior regularity of solutions of elliptic equations, $v' \in W^{2m,2}_{\text{loc}}(G'')$ and $Av' = w - v$ in G''. But v' and $w - v$ both are zero outside of G in G'', so that by the argument of Lemma 10, v' is the limit in $W^{2m,2}(G'')$ of functions from $C_c^{2m}(G \cup \Gamma_0)$. In particular therefore, v' satisfies the null Cauchy boundary conditions of order $2m$ on Γ_1 and lies in $D(A_{2,0})$. It follows that $v \in D(A_{2,0}A'_{2,1})$ and $(A_{2,0}A'_{2,1} + I)v = w$. Thus T_2 is onto, and Theorem 5 is proved.

The remainder of Section 3 is devoted to extending the conclusions of Theorem 5 to the case in which $p \neq 2$. We carry through this extension under either one of a set of two alternative additional hypotheses, the first being a mild extra local regularity assumption on the coefficients of A, and the second being the assumption that the resolvent set of A_2 is non-empty. In Section 4, we shall show that the latter hypothesis is verified for all uniformly strongly elliptic operators.

An important role in the subsequent discussion is played by the following result from the regularity theory of the Dirichlet problem for elliptic equations with constant coefficients:

Lemma 15. *Let B_0 be the unit ball in E^n, $B_0 = \{y : |y| < 1\}$ $B_0^+ = B_0 \cap \cap \{y : y_1 > 0\}$, $P_0 = B_0 \cap \{y : y_1 = 0\}$. Let A_0 be a homogeneous regularly elliptic differential operator of order $2m$ with constant coefficients in E^n. Then:*

(i) *There exists a bounded linear mapping T_0 of $L^p(B_0)$ into itself such that for all u in $L^p(B_0)$, $T_0 u \in W^{2m,p}_{\text{loc}}(B_0)$ and $A_0 T_0 u = u$ in B_0.*

(ii) *There exists a bounded linear mapping T of $L^p(B_0^+)$ into itself such that for each u in $L^p(B_0^+)$, Tu lies in $W^{2m,p}_{\text{loc}}(B_0^+ \cup P_0)$, $A_0 Tu = u$ in B_0^+, and Tu satisfies the null Dirichlet conditions of order m on P_0.*

(iii) *$\|T\|, \|T_0\|$ are uniformly bounded by a fixed multiple of the ellipticity bounds on A_0.*

Remark. The continuity of the maps T_0 and T implies that they are closed mappings of $L^p(B_0)$ and $L^p(B_0^+)$, respectively, into $W^{2m,p}_{\text{loc}}(B_0)$ and $W^{2m,p}_{\text{loc}}(B_0^+ \cup P_0)$, where each of the latter is considered in the natural way as a Frechet space. Indeed suppose $u_n \to u$ in $L^p(B_0)$ $(L^p(B_0^+))$ while $T_0 u_n (Tu_n)$ converges to v in $W^{2m,p}(C)$ for every compact subset C of $B_0(B_0^+ \cup P_0)$. Since $T_0 u_n (Tu_n)$ converges also in $L^p(B_0)$ $(L^p(B_0^+))$ to $T_0 u(Tu)$, it follows immediately that $v = T_0 u$ or Tu, respectively, and that T_0 and T are closed mappings in the appropriate spaces. By the closed graph theorem, however, it follows that for any compact subset C of $B_0(B_0^+ \cup P_0)$, there exists a constant $K(C)$ such that $\|T_0 u\|_{W^{2m,p}(C)} \leqq K(C)\|u\|_{L^p(G)}$, $\|Tu\|_{W^{2m,p}(C)} \leqq K(C)\|u\|_{L^p(G)}$.

Proof of Lemma 15. The result follows from the corresponding results for the whole of E^n and for the half-space $\{y : y_1 > 0\}$, obtained in [16] and [17],

and in the latter case from an integral representation of the solution of the Dirichlet problem for $Au = f$ with null Dirichlet boundary conditions of order m on the plane $\{y : y_1 = 0\}$. (The fact that u lies in $W^{2m,p}_{\text{loc}}$ if f lies in L^p follows from the integral inequalities of CALDERON and ZYGMUND [20], or as in [21], from similar results on multipliers of multiple Fourier series due to MARCINKIEWICZ [46].)

Theorem 6. *Let G be an open subset of E^n with G uniformly regular of class C^{2m}, A a regularly elliptic differential operator defined on G and satisfying the regularity conditions* (I) *and* (II) *of Section* 1 *on G. Let G' be a bounded subdomain of G, Γ_0, the intersection of the boundary of G' with the boundary Γ of G. Let $A''_{p,1}$ for $1 < p < \infty$ be the realization of A in $L^p(G')$ with $D(A''_{p,1}) = \{u : u \in W^{2m,p}_{\text{loc}}(G' \cup \Gamma_0) \cap L^2(G'), Au \in L^2(G'), D^\beta u | \Gamma_0 = 0 \text{ for } |\beta| < m\}$. Then the range of $A''_{p,1}$ is a closed subspace of finite co-dimension in $L^p(G')$. In particular, if G is bounded and $G' = G$, the range of A_p is a closed sub-space of finite co-dimension in $L^p(G)$.*

Proof of Theorem 6. We begin by considering the covering $\{N_k\}$ of G defined in Lemma 2 for some given $\delta < \delta_0$. If A_y is the result of transplanting A by the homeomorphism Φ_k, it follows from Lemma 2(b), since the bound K for the moduli of the components of Φ_k and its inverse Ψ_k are independent of δ, that the homogeneous operator A_0 with constant coefficients obtained by evaluating the coefficients of A_y at a fixed point is uniformly elliptic on E^n with the constant of ellipticity and the bounds on its coefficients independent of δ. It follows then from Theorem 2′ of Section 2 that if $u = Tf$, $u_0 = T_0 f$, where T and T_0 are defined as in Lemma 15 for any such A_0, with f lying in $L^p(B_0^+)$ or $L^p(B_0)$, respectively, that

$$\|Tf\|_{W^{2m,p}(C)} \leqq K(C)\{\|f\|_{L^p(B_0^+)} + \|Tf\|_{L^p(B_0^+)}\}, \tag{3.14}$$

for any open subset C of B_0^+ with compact closure in $B_0^+ P_0$, and

$$\|T_0 f\|_{W^{2m,p}(C)} \leqq K(C)\{\|f\|_{L^p(B_0)} + \|T_0 f\|_{L^p(B_0)}\}, \tag{3.15}$$

for any open subset C of B_0 with compact closure in B_0. By (iii) of Lemma 15, however, $\|T\|$ and $\|T_0\|$ as mappings of their corresponding L^p-spaces are uniformly bounded when the constant of ellipticity and the bounds on the coefficients of A_0 are uniform. It follows therefore that with a suitable change of the coefficient $K(C)$, we have

$$\|Tf\|_{W^{2m,p}(C)} \leqq K(C)\|f\|_{L^p(B_0^+)}, \tag{3.16}$$

$$\|T_0 f\|_{W^{2m,p}(C)} \leqq K(C)\|f\|_{L^p(B_0)}. \tag{3.17}$$

If we let T_δ and $T_{0,\delta}$ be the mappings of $L^p(B_\delta^+)$ and $L^p(B_\delta)$, respectively, into themselves defined by composing T and T_0 with the injection mappings of $L^p(B_\delta)$ into $L^p(B_0)$ and of $L^p(B_\delta^+)$ into $L^p(B_0^+)$ on the right and the restriction mappings going in the opposite direction on the left, i.e. $T_\delta f = Tf|_{B_\delta}$ with f set equal to zero outside of B_δ, then T_δ and $T_{0,\delta}$ satisfy (3.16) and (3.17), with constants $K(C)$ independent of δ or C for each C with compact closure in B_δ or B_δ^+, $\delta < 1$.

Let $A_y = \sum_{|\alpha| \leq 2m} a_\alpha(y) D_y^\alpha$ be the result of transplanting A from one of the sets $N_k \cap G$ into B_δ (or B_δ^+) $A_0 = \sum_{|\alpha| = 2m} a_\alpha(0) D^\alpha$. It follows from the continuity assumption on the top order terms of A that for δ sufficiently small, the sum $\sum_{|\alpha| = 2m} |a_\alpha(y) - a_\alpha(0)|$ can be made as small as we please on $B_\delta(B_\delta^+)$. Let f be an arbitrary element of $L^p(B_\delta)$, and consider the problem of finding g in $L^p(B_\delta)$ such that $A_y T_{0,\delta} g = f$. Since $A_y = A_0 + (A_y - A_0)$, we see that

$$A_y T_{0,\delta} g = A_0 T_{0,\delta} g + (A_y - A_0) T_{0,\delta} g = g + (A_y - A_0) T_{0,\delta} g .$$

Let $Rg = (A_y - A_0) T_{0,\delta} g$. Then, $Rg = \sum_{|\alpha| = 2m} (a_\alpha(y) - a_\alpha(0)) D^\alpha T_{0,\delta} g + \sum_{|\alpha| < 2m} a_\alpha(y) D^\alpha T_{0,\delta} g$, and we obtain the inequality

$$\|Rg\|_{L^p(B_\delta)} \leqq \varepsilon \|T_{0,\delta} g\|_{W^{2m,p}(B_\delta)} + K \|T_{0,\delta} g\|_{W^{2m-1,p}(B_\delta)} . \tag{3.18}$$

In (3.18), ε can be made less than $(4K(B_\delta))^{-1}$ by choosing δ sufficiently small, while on B_δ, $\|u\|_{W^{2m-1,p}(B_\delta)} \leqq K\delta \|u\|_{W^{2m,p}(B_\delta)}$, (the latter by a dimensionality argument on the derivatives of different orders). In (3.16), since $K(C)$ is independent of C, we may choose C to be B_δ itself and obtain

$$\|Tg\|_{W^{2m,p}(B_\delta)} \leqq K(B_\delta) \|g\|_{L^p(B_\delta)} . \tag{3.19}$$

Putting (3.19) into (3.18) and using the bound on the $W^{2m-1,p}(B_\delta)$ norm in terms of the $W^{2m,p}(B_\delta)$-norm, we obtain

$$\|Rg\| \leqq \{4^{-1} + K_1 \delta\} \|g\| . \tag{3.20}$$

Choosing δ sufficiently small makes $\|R\| < 2^{-1}$. On the other hand, $A_y T_{0,\delta} g = g + Rg = f$. It follows that $g = (I + R)^{-1} f$ exists for every f, and the new mapping $T'_{0,\delta} = T_{0,\delta}(I + R)^{-1}$ gives a mapping of $L^p(B_\delta)$ into itself with the image of each f lying in $W^{2m,p}(B_\delta)$ and satisfying the equation $A_y T'_{0,\delta} f = f$.

By a parallel argument, if δ is sufficiently small we construct a mapping T' of $L^p(B_\delta^+)$ into itself, with $T'_\delta f$ lying in $W^{2m,p}(B_\delta^+)$, satisfying the null Dirichlet boundary conditions of order m on $\{y : y_1 = 0\}$, and having $A_y T'_\delta f = f$.

Let $\{\eta_k(x)\}$ be the sequence of functions corresponding to the covering $\{N_k\}$ and $\delta > 0$, as in Lemma 2 of Section 1. If $f \in L^p(G)$, we define $S_k f(x)$ by

$$(S_k f)(x) = \eta_k(x) (T'_{0,\delta} f_k)(\Phi_k(x)) , \tag{3.21}$$

with $f_k(y) = (\eta_k f)(\Psi_k(y))$, for those N_k which do not intersect Γ, and

$$(S_k f)(x) = \eta_k(x) (T'_\delta f_k)(\Phi_k(x)), f_k(y) = (\eta_k f)(\Psi_k(y)) , \tag{3.22}$$

for those N_k which do intersect Γ. Since for each k, f_k lies in $L^p(B_\delta)$ or $L^p(B_\delta^+)$ respectively, it follows from (3.21) and (3.22) and the previously discussed properties of the transformations T'_δ and $T'_{0,\delta}$ that $S_k f$ lies in $W^{2m,p}(N_k \cap G)$ for every k, and indeed may be extended to an element of $W^{2m,p}(G)$ by setting it equal to zero in $G - (N_k \cap G)$. If N_k intersects Γ, it follows from the properties of T'_δ that $S_k f$ satisfies the null Dirichlet boundary conditions of order m on $\Gamma \cap N_k$ and, since it vanishes identically on a neighborhood of all other points

of Γ, it satisfies the null Dirichlet boundary conditions of order m on all of Γ. The same is true for the N_k which do not meet Γ since in that case $S_k f$ has compact support in G.

On the other hand, (if $N_k \cap \Gamma \neq \Theta$)

$$(A S_k f)(x) = A\left(\eta_k T'_\delta f_k(\Phi_k(x))\right) = \eta_k(x)\, A\{T'_\delta f_k(\Phi_k(x))\} + \sum_{|\beta|<2m} c_{\beta,k}(x)\, D^\beta\{T'_\delta f_k(\Phi_k(x))\}\,. \tag{3.23}$$

For the first term of the last sum,

$$\begin{aligned}\eta_k(x)\, A\{T'_\delta f_k(\Phi_k(x))\} &= \eta_k(x)\,(A_y T' f_k)(\Phi_k(x)) \\ &= \eta_k(x)\, f_k(\Phi_k(x)) = \eta_k(x)\,\eta_k(x)\, f(x)\,.\end{aligned} \tag{3.24}$$

A similar calculation for the case when $N_k \cap \Gamma = \Theta$ yields the following equality for all k,

$$(A S_k f)(x) = \eta_k^2(x)\, f(x) + \sum_{|\beta|<2m} c_{\beta,k}(x)\, D^\beta S'_{\beta,k} f(x)\,, \tag{3.25}$$

where $S'_{\beta,k}$ is a bounded linear mapping of $L^p(G)$ into $W^{2m,p}(G)$.

Now consider all the neighborhoods N_k of the given covering which intersect G', and let Sf for f in $L^p(G')$ be defined by

$$Sf = \sum_k S_k f\,, \tag{3.26}$$

where, as usual, f is set equal to zero outside G' and the sum is taken over the finite family of k for which $N_k \cap G'$ is not vacuous. It follows from our preceeding remarks that Sf lies in $W^{2m,p}(G)$ and hence in $W^{2m,p}(G')$, Sf satisfies the null Dirichlet boundary conditions of order m at all points of Γ_0, and finally $A Sf = \sum_k A\, S_k f = \sum_k \eta_k^2 f + \sum_k \sum_{|\beta|<2m} c_{\beta,k}(x)\, D^\beta(S'_{\beta,k} f)(x)$. Thus

$$A Sf = f + Rf\,, \tag{3.27}$$

where by Lemma 5(b), since G' is bounded, R is a compact linear operator in $L^p(G')$.

For each f in $L^p(G')$, Sf obviously lies in the domain of $A''_{p,1}$ and thus the range of $A''_{p,1}$ is a linear subset of $L^p(G')$ which contains the range of $(I + R)$, R compact. But, the range of $(I + R)$ by a classical theorem of F. RIESZ is a closed subspace of finite co-dimension in $L^p(G')$. Therefore, the range of $A''_{p,1}$, which contains a closed subspace of finite co-dimension, must by an elementary argument be itself a closed subspace of $L^p(G)$ with finite co-dimension and Theorem 6 is proved.

To treat adjointness relations in L^p for the general elliptic operator, we must consider some local regularity conditions for the coefficients. (For the strongly elliptic case, our later discussion yields the corresponding results without these further assumptions, as we shall see in the last part of the present Section.)

(V) *The coefficients $a_{\alpha,k}$ of A and $a'_{\alpha,k}$ of A' are in C^1 in a neighborhood of each point of $G \cup \Gamma$ for $|\alpha| = 2m$, and satisfy local Lipschitz condition of order one in the neighborhood of each point of $G \cup \Gamma$ for general α.*

(U) *A' will be said to have uniqueness of the Cauchy problem in the small if A' has coefficients satisfying* (V) *and if for a connected domain G of E^n, every solution v in $C^{2m}(G)$ of the equation $A'v = 0$ which vanishes on an open subset of G must vanish identically in G.*

Remark. CALDERON ([19]) has established the fact that A' has property (U) if the characteristic form of A has only simple roots in the complex domain, a result which generalizes earlier work of CARLEMAN for $n = 2$ and other writers for second-order equations. (See the introduction and bibliography of [19] for a discussion of the history of this problem.) Whether (U) holds for all elliptic operators is still an open question at the time of writing[2]). We emphasize the fact that (U) is a local condition and has no relation to the behaviour of the coefficients of A in the large.

Remark. The conditions (V) and (U) obviously hold for operators A whose coefficients are analytic on E^n, the second from Holmgren's theorem (JOHN [37]) or from the less elementary fact of the analyticity of solutions of elliptic equations with analytic coefficients.

Theorem 7. *Let G be a connected open subset of E^n, with G uniformly regular of class C^{2m} and locally regular of class C^{4m}. Let A be a regularly elliptic operator on G which satisfies the regularity conditions* (IV) *and* (V), *while A' satisfies condition* (U). *Let G' be a bounded open subset of G such that G is not contained in the closure of G'. Let $A''_{p,1}$ be the operator in $L^p(G')$ defined in Theorem* 6. *Then the range of $A''_{p,1}$ is the whole of $L^p(G')$.*

Before proceeding to the proof of Theorem 7, we shall give the proof of the following Theorem 8, whose conclusion is applied in the proof of Theorem 7 as well as in the subsequent discussion.

Theorem 8. *Let G be an open subset of E^n, with G uniformly regular of class C^{2m} and locally regular of class C^{2m+1}. Let A be a regularly elliptic operator on G which satisfies the regularity conditions* (I) *and* (II) *of Section* 1. *Suppose the boundary Γ of G is divided into two parts Γ_0 and Γ_1 which satisfy the regularity conditions of Lemma* 10, *and let $A_{p,1}$ be the realization of A in $L^p(G)$ for $1 < p < \infty$ satisfying the null Dirichlet boundary conditions on Γ_0. Let $f \in R(G)$, $p \leqq p_1 < \infty$.*

Suppose that for all j and α, $D_j a_\alpha$ is locally essentially bounded on a neighborhood of each point of $G \cup \Gamma$.

(a) *Let u be a function from $D(A_{p,1})$ such that $Au = f$. Then u for $\lambda < 1$, lies in $C^{2m,\lambda}_{\mathrm{loc}}(G \cup \Gamma_0)$, where $C^{2m,\lambda}_{\mathrm{loc}}(G \cup \Gamma_0)$ is the space of functions whose derivatives of order $2m$ satisfy a Hölder condition with exponent λ on a neighborhood in G of each point of $G \cup \Gamma_0$.*

(b) *Let u be a function from $D(A_p)$ such that $Au = f$. Then for any $\lambda < 1$, u lies in $D(A_{p_1}) \cap C^{2m,\lambda}(G)$, where $C^{2m,\lambda}(G)$ is the space of functions whose derivatives of order $2m$ satisfy a uniform Hölder condition on G.*

Proof of Theorem 8. We begin with the proof of (b). If u lies in $D(A_p)$ and $Au = f$, where $f \in R(G) \subset W^{1,p}(G)$, it follows from Theorem 4 that u lies in

[2]) As remarked in the Introduction, this question has been settled in the negative by the examples of COHEN and PLIS.

$W^{2m+1,p}(G)$. By Lemma 5(a), however, $W^{2m+1,p}(G) \subset W^{2m,r}(G)$ for any r with $p \leqq r \leqq np(n-p)^{-1}$ [3]). Hence $u \in D(A_r)$ for such r, and proceeding step by step, $u \in D(A_r)$ for all $r \geqq p$, by Theorem 4 and $u \in W^{2m+1,r}(G)$. The conclusion about the Hölder continuity of $D^\alpha u$ for $|\alpha| \leqq 2m$ follows from this last fact and Lemma 5(c) and the proof of (b) is complete.

To prove (a), we first remark that if $u \in D(A_{p,1})$ and $Au \in W^{1,p}_{\text{loc}}(G \cup \Gamma_0)$, then $u \in W^{2m+1,p}_{\text{loc}}(G \cup \Gamma_0)$. Indeed, let C be a compact subset of $G \cup \Gamma_0$, $\zeta(x)$ a function from $R(G)$ whose support intersects G in a compact subset of $G \cup \Gamma_0$ and which is identically 1 on C. Then ζu lies in $D(A_p)$, and $A(\zeta u) = \zeta Au + \sum_{|\beta|<2m} c_\beta(x) D^\beta u$ obviously lies in $W^{1,p}(G)$. By Theorem 4, it follows that ζu must lie in $W^{2m+1,p}(G)$, and since $\zeta = 1$ on C, we have $u \in W^{2m+1,p}(C)$, and thus $u \in W^{2m+1,p}_{\text{loc}}(G \cup \Gamma_0)$. We now apply the same argument as for (b), using the above remark instead of Theorem 4. If u lies in $D(A_{p,1})$ with $Au = f$, and $f \in R(G) \subset W^{1,p}(G)$, it follows that u lies in $W^{2m+1,p}_{\text{loc}}(G \cup \Gamma_0)$. From the local version of Lemma 5 we see that u must lie in $W^{2m,r}_{\text{loc}}(G \cup \Gamma_0)$ for $p \leqq r \leqq np(n-p)^{-1}$, and hence $\zeta u \in D(A_{r,1})$ for such r. Proceeding step by step, $\zeta u \in D(A_{r,1})$ and $u \in W^{2m+1,r}_{\text{loc}}(G \cup \Gamma_0)$ for all $r \geqq p$. The conclusion that u lies in $C^{2m,\lambda}_{\text{loc}}$ follows from the local version of Lemma 5(c), and the proof of Theorem 8 is complete.

Proof of Theorem 7. Since G' is bounded, $D(A''_{2,1}) \subset D(A''_{p,1})$ for any $p < 2$. If we prove Theorem 7 for $p = 2$, it will follow that the range of $A''_{p,1}$ will contain the range of $A''_{2,1}$ and in particular the dense subset $R(G')$ of $L^p(G')$. Since the range of $A''_{p,1}$ is closed in $L^p(G')$ by Theorem 6, it would follow that it must be the whole of $L^p(G')$. Thus it suffices to consider only $p \geqq 2$. On the other hand, suppose $u \in D(A''_{2,1})$ is a solution of the equation $Au = f$, $f \in R(G)$, on a domain G'' such that $d(G', G - G'') > 0$, with u satisfying null Dirichlet boundary conditions on the intersection Γ_2 of the boundaries of G and G''. Applying Theorem 8(a), or rather its proof, (since the statement of the Theorem demands un-necessarily that G'' itself be an uniformly regular domain) it follows that $u \in W^{2m,p}_{\text{loc}}(G'' \cup \Gamma_2)$ for all $p > 2$. But then $u \in W^{2m,p}(G')$, u must lie in $D(A''_{p,1})$, and $Au = f$ in G'. The range of $A''_{p,1}$ being dense since it contains the dense subset $R(G')$ of $L^p(G')$ and being closed by Theorem 6, it would follow that it must be the whole of $L^p(G')$. It suffices therefore to establish Theorem 7 for the case in which $p = 2$.

Suppose that the range of $A''_{2,1}$ is not the whole of $L^2(G')$. Since it is closed by Theorem 6, there must exist a non-null element v of $L^2(G')$ such that $(Au, v) = 0$ for all u in $D(A''_{2,1})$. Setting $v = 0$ outside of G' and noting that the restriction of any function u in $D(A_{2,1})$ to G' lies in $D(A''_{2,1})$, it follows that $(Au, v) = 0$ for all u in $D(A_{2,1})$. The latter statement is equivalent to the assertion that v lies in $D((A_{2,1})^*)$ and that $A^*_{2,1}v = 0$. Since the operator A satisfies the regularity condition (IV), Theorem 5 states that $(A_{2,1})^* = A'_{2,0}$. Thus v lies in $D(A'_2)$ and $A'v = 0$. By the regularity condition (V) on A, the

[3]) In the boot-strap arguments on the exponent, the upper bound on the inequalities should be $< \infty$ if $n - p = 0$, and $\leqq \infty$ if $n - p < 0$.

differential operator A' satisfies the conditions on A imposed in Theorem 8. By the conclusion of that theorem, v must lie in $C^{2m,\lambda}(G)$ for all $\lambda < 1$, while by our previous remarks $A'v = 0$, $v = 0$ on $G - G'$. Since G is connected, $G - G'$ has an interior, and A' is supposed to have uniqueness of the Cauchy problem in the small, it follows that $v = 0$, contradicting the assumption that the range of $A''_{2,1}$ is not the whole of $L^2(G')$. Thus, the decisive case of Theorem 7 and hence Theorem 7 itself is completely established.

Using Theorems 7 and 8, we now give a general adjointness theorem in L^p for the general elliptic operator.

Theorem 9. *Let G be an open subset of E^n, with G uniformly regular of class C^{2m} and locally regular of class C^{4m}. Let A be a regularly elliptic operator on G satisfying the regularity conditions* (IV) *and* (V), *A' its formal adjoint. Suppose that A and A' both satisfy condition* (U). *Let Γ_0 be a subset of the boundary Γ of G, with Γ_0 regular in the sense of Lemma* 10, *$\Gamma_1 = \Gamma - \Gamma_0$. For $1 < p < \infty$, $q = p(p-1)^{-1}$, let A_p, $A_{p,0}$, and $A_{p,1}$ be the realizations of A in $L^p(G)$ defined in Section* 2 *with respect to the subset Γ_0 of Γ; A'_q, $A'_{q,0}$, and $A'_{q,1}$ the corresponding realizations of A' in $L^q(G)$.*

Then, $A_p^* = A'_q$, $A_{p,0}^* = A'_{q,1}$, and $A_{p,1}^* = A'_{q,0}$.

Proof of Theorem 9. For $p = q = 2$, the conclusion of Theorem 9 is contained in Theorem 5. By Lemma 14, we know that $A'_q \subseteq A_p^*$, $A'_{q,1} \subseteq A_{p,0}^*$, and $A'_{q,0} \subseteq A_{p,1}^*$. We need therefore only prove inclusion in the reverse direction. Note that A_p may be considered as a special case of $A_{p,0}$ and $A_{p,1}$ when $\Gamma_0 = \Gamma$.

All the operators considered are closed and the conditions of the Theorem are symmetric in A and A'. If we prove that the Theorem holds for $p \leqq 2$, then, applying this result to A' and to the exponent q if $p < 2$, we have $(A'_{q,0})^* = A_{p,1}$ and $(A'_{q,1})^* = A_{p,0}$, and taking adjoints, we obtain $(A_{p,1})^* = A'_{q,0}$ and $(A_{p,0})^* = A'_{q,1}$. We may restrict ourselves without loss of generality to the case $p \leqq 2$, and correspondingly $q \geqq 2$.

Suppose $v \in D(A_{p,1}^*)$. Then for f in $R(G) \cap D(A_{p,1})$, we have $(Af, v) = (f, g)$ where g is a function from $L^q(G)$. To prove that v lies in $D(A'_{q,0})$, it suffices to show that v lies in $D(A'_q)$, since then the Cauchy boundary conditions on Γ_1 follow from the equality $(Af, v) = (f, A'v)$, $(f \in D(A_{p,1}))$ by the argument of Lemma 12. On the other hand, since $(A_{p,1})^* \subseteq A_p^*$, v already lies in $D(A_p^*)$. Thus it suffices to show that $D(A_p^*) = D(A'_q)$. To show that an element v from $D(A_p^*)$ lies in $D(A'_q)$, however, it suffices by Lemma 12 and Theorem 3 to show that v lies in $W^{2m,q}(N)$ for some neighborhood N of each point of $G \cup \Gamma$. If x_0 is a point of Γ, let Γ_0 be a regular piece of Γ containing x_0, and consider the operators $A_{p,0}$ and $A'_{q,1}$ defined with respect to Γ_0. Obviously, v lies in $D(A_{p,0}^*)$, and if we show that it lies in $D(A'_{q,1})$, it will follow that v lies in $W^{2m,q}(N)$ for some neighborhood N of x_0 in G. Thus we have reduced the proof of Theorem 9 to the special case of proving that $D(A_{p,0}^*) = D(A'_{q,1})$ with $p \leqq 2$, $q \geqq 2$. We may reduce it still further by noting that to prove that a function v from $D(A_{p,0}^*)$ lies in $D(A'_{q,1})$, it suffices to show that v lies in $W^{2m,p}(N)$ for some neighborhood N of each point of $G \cup \Gamma_0$, that this neighborhood N may

itself be taken as uniformly regular of class C^{4m} bounded, and connected, and that v restricted to N lies in $D(A^*_{p,0,N})$, where the operator $A_{p,0,N}$ is defined on $L^p(N)$ in the same way as $A_{q,0}$ was defined on G, with respect to the portion Γ''_0 of the boundary Γ'' of N which consists of $\Gamma'' \cap \Gamma_0$. Thus after replacing G by N, we may assume that G is bounded and connected, as well as satisfying the previous regularity conditions.

For this last and simplest case, we remark that since $q \geqq 2$ and G is bounded, v lies in $L^2(G)$, and satisfies the relation

$$(A f, v) = (f, g)\,, \tag{3.28}$$

for all f in $D(A_{2,0}) \cap R(G)$ with a given g in $L^q(G)$. Since G is bounded, g also lies in $L^2(G)$. By Lemma 10, the restriction of $A_{2,0}$ to the subset $R(G) \cap A_{2,0}$ of its domain is an operator which has $A_{2,0}$ as its closure. Thus, v lies in $D(A^*_{2,0})$, which by Theorem 5 equals $D(A'_{2,1})$, v lies in $W^{2m,2}_{\text{loc}}(G \cup \Gamma_0)$ and $A'v = g$, with g in $L^q(G)$. We seek to prove that v lies in $W^{2m,q}_{\text{loc}}(G \cup \Gamma_0)$. We may assume from our preceding construction that Γ_0 is not the complete boundary of G and that G may be included in a large regular domain G_1 whose boundary contains Γ_0 as a proper part. If we consider the operator $A'_{q,1}$ in the guise of the operator studied in Theorem 7 with A replaced by A' and G considered as a subdomain of G_1, we see that $A'_{q,1}$ maps, onto $L^q(G)$. Thus there exists w in $D(A'_{q,1})$ such that $A'w = g$. Setting $v_1 = v - w$, we have $A'v_1 = 0$, while $v_1 \in D(A'_{2,1})$. By Theorem 8(a), however, v_1 has its $2m$-th derivatives continuous on $G \cup \Gamma_0$ and hence lies in $W^{2m,q}_{\text{loc}}(G \cup \Gamma_0)$. Hence so does v, and the proof of Theorem 9 is complete.

To obtain the conclusion of Theorem 9 without the special regularity assumptions on the coefficients of A and A' (as well as the assumption of uniqueness of the Cauchy problem in the small), we shall use the following theorem:

Theorem 10. *Let G be an open subset of E^n which is uniformly regular of class C^{2m} and locally regular of class C^{2m+1}. Let A be a regularly elliptic operator of order $2m$ on G which satisfies the regularity conditions* (I) *and* (II) *on G. Suppose that for every bounded open subset G' of G which is uniformly regular of class C^{4m} that $A^0_{2,(G')}$ has a non-empty resolvent set, where $A'_{2,(G')}$ is the realization of the top order homogeneous part of A in $L^2(G')$ under null Dirichlet conditions of order m on the boundary of G'. Let $2 \leqq p \leqq p_1 < \infty$. Then if u lies in $D(A_p)$ with $Au \in L^{p_1}(G)$, it follows that $u \in D(A_{p_1})$.*

Proof of Theorem 10. To prove that u lies in $D(A_{p_1})$, it suffices by Theorem 3 to prove that u lies in $W^{2m,p_1}(N)$ for some neighborhood N of each point of $G \cup \Gamma$.

Indeed, since u lies in $W^{2m,p}(G)$, it follows that u lies in $L^r(G)$ for $p \leqq r \leqq np(n-p)^{-1}$ [3]. If p_1 lies in this range, then u lies in $L^{p_1}(G)$, already. For larger p_1, we must progress by a step-by-step argument, remarking that since u lies in $W^{2m,p}(G)$, Au lies in $L^p(G)$, while by hypothesis Au lies in $L^{p_1}(G)$. Hence if $p_1 \geqq np(n-p)^{-1} = r$, Au lies in $L^r(G)$ by the usual convexity argument for L^s-norms, and if we apply the argument above for $p = r$, it

follows since u lies in $L^r(G)$, $Au \in L^r(G)$, that u lies in $W^{2m,r}(G)$. We now resume the argument replacing the assumption that u lies in $W^{2m,p}(G)$, (or more precisely, that u lies in $D(A_p)$) by our conclusion that u lies in $W^{2m,r}(G)$, (or more precisely, that u lies in $D(A_r)$, which obviously follows). If the first step of the argument was valid for all p_1 with $p^{-1} - n^{-1} \leqq p_1^{-1} \leqq p^{-1}$, the second step will be valid for all p_1 with $p^{-1} - 2n^{-1} \leqq p_1^{-1} - n^{-1}$, and, more generally, proceeding j steps, the j-th step will be valid for all p_1 with $p^{-1} - jn^{-1} \leqq p_1^{-1} \leqq p^{-1} - (j-1)n^{-1}$. But for $j > np^{-1}$, the j-th step, together with its predecessors, will include all $p_1 \geqq p$. We shall apply the same argument below without a repetition of the details, successively increasing the range of p_1 for which the result under consideration is valid by use of Sobolev's theorem (Lemma 5), and we shall refer to it for the sake of brevity as the boot-strap argument. (We remark that there are a number of distinct boot-strap arguments of a type similar to the given one which are current in the theory of partial differential operators, boot-strapping on order of differentiability, etc., but it will be clear from the summary description given which process we are applying.)

Let $\lambda(x)$ be a function from $R(G)$ which has its support in a neighborhood N_1 of a given point x_0 of $G \cup \Gamma$ (where we may assume that N_1 is uniformly regular of class C^{2m+1} and bounded) with $\lambda(x) = 1$ on a neighborhood N of x_0. The function $v = \lambda u$ lies in $W^{2m,p}(G)$ and indeed in $D(A_p)$, the support of v is contained in N_1, and

$$Av = A(\lambda(x)u) = \lambda(x)\,Au + \sum_{|\beta|<2m} c_\beta(x) D^\beta u\,, \tag{3.29}$$

where the functions $c_\beta(x)$ are uniformly bounded and have their supports in N_1. From the fact that u lies in $W^{2m,p}(G)$ and Lemma 5, it follows that the last sum in (3.29) is an element of $L^r(G)$ for any r for which $p \leqq r \leqq np(n-p)^{-1}$ [3]). If we can establish the Theorem for all p_1 in this range, then repeating the argument starting with p_1 instead of p and proceeding step by step, we should establish our result for v for all $p_1 \geqq p$ and hence for u. Thus it suffices to assume that $p \leqq p_1 \leqq np(n-p)^{-1}$ and that u has its support in a bounded domain G which is uniformly regular of class C^{2m+1}.

Since u lies in $W^{2m,p}(G)$ and $p_1 \leqq np(n-p)^{-1}$, it follows from Lemma 5 that for $|\beta| < 2m$, $D^\beta u$ lies in $L^{p_1}(G)$. By assumption, there exists a complex number ζ such that $(A_2^0 - \zeta I)$ is a invertible mapping onto $L^2(G)$, where A^0 is the top-order homogeneous part of A. Since $(A^0 - \zeta I)u = Au - Su - \zeta u$, where S consists of all terms of lower order in A, it follows that $(A^0 - \zeta I)u$ lies in $L^{p_1}(G)$. Hence we lose no generality in assuming that A_2 is an invertible mapping onto $L^2(G)$ and that the coefficients of A are uniformly continuous on G.

Following a similar line of argument to that of Lemma 13 (but under slightly weaker regularity conditions) we approximate A on G by a sequence $A^{(k)}$ of differential operators with analytic coefficients on E^n, so that if $A^{(k)} = \sum_{|\alpha| \leqq 2m} a_\alpha^{(k)}(x) D^\alpha$, the coefficients $a_\alpha^{(k)}(x)$ converge uniformly on G to the coefficients a_α of A. It follows immediately, (since $A_2^{(k)} = A_2 + (A_2^{(k)} - A_2)$

$= \{I + (A_2^{(k)} - A_2)A_2^{-1}\} A_2$, while by Theorem 2 A_2^{-1} is a bounded mapping of $L^2(G)$ into $W^{2m,2}(G)$ provided it is a bounded mapping into $L^2(G)$), that for sufficiently large k, $A_2^{(k)}$ is an invertible mapping onto $L^2(G)$, $\|(A_2^{(k)})^{-1}\| \leqq M$ for all large k, while $(A_2^{(k)})^{-1}v$ converges to $(A_2^{-1}v)$ strongly in $W^{2m,2}(G)$ as $k \to \infty$.

Suppose now that $p = 2$, and that $g \in L^{p_1}(G)$, with $p_1 \leqq 2n(n-2)^{-1}$. By Theorem 8, $(A_2^{(k)})^{-1}g$ lies in $W^{2m,p_1}(G)$, and if we set $u^{(k)} = (A_2^{(k)})^{-1}g$, by Theorem 2, we have

$$\|u_k\|_{W^{2m,p_1}(G)} \leqq K\{\|g\|_{L^{p_1}(G)} + \|u_k\|_{L^{p_1}(G)}\}. \tag{3.30}$$

By Lemma 5, however, $\|u_k\|_{L^{p_1}(G)} \leqq K_1\|u_k\|_{W^{2m,2}(G)} \leqq K_2$ for all k, so that the u_k form a bounded sequence in $W^{2m,p_1}(G)$. Choosing a subsequence, we may assume that u_k converges weakly in $W^{2m,p_1}(G)$ to an element u_0 of $W^{2m,p_1}(G)$. But weak convergence in $W^{2m,p_1}(G)$ implies strong convergence in $L^2(G)$ to the same limit, while we already know that the u_k converge to the given function u. Hence u lies in $W^{2m,p_1}(G)$, and the case in which $p = 2$ is complete. Let us remark moreover that for p_1 in the restricted range $2 \leqq p_1 \leqq 2n(n-2)^{-1}$, A_{p_1} is an invertible operator mapping onto $L^{p_1}(G)$. Indeed any element of the null space of A_{p_1} would lie in the null space of A_2 (since G is bounded) and the latter null-space is trivial. On the other hand, we have shown that for g in $R(G)$, $A_2^{-1}g$ lies in $D(A_{p_1})$ and therefore the range of A_{p_1} is dense. Since by Theorem 6, the range of A_{p_1} is closed in A_{p_1}, A_{p_1} must be a one-to-one mapping of its domain onto $L^{p_1}(G)$, and is invertible by the closed graph theorem.

To complete the proof, we proceed step by step, using the invertibility of A_{p_1} as established in the preceding step, using the above argument with the exponent 2 replaced by any other p for which the invertibility of A_p has been established. The whole range of exponents $p_1 \geqq p$ is covered by this procedure in a finite number of steps.

Remark: The fact that the range of A_p is closed, as used in the preceding proof, can be obtained directly from Theorem 2, without the use of the argument of Theorem 6. For later reference, we shall give this proof.

Theorem 11. *Let G be a domain in E^n uniformly regular of class C^{2m}, A a regularly elliptic operator on G satisfying the regularity conditions* (I) *and* (II) *of Section* 1, $1 < p < \infty$. *Then if G is a bounded domain, the null space of A_p is of finite dimension, and the range of A_p is a closed subspace of $L^p(G)$.*

Proof of Theorem 11. For u in the null space of A_p, it follows from Theorem 2 that $\|u\|_{W^{2m,p}(G)} \leqq K\|u\|_{L^p(G)}$. If we consider the null space N_p as a linear subset of $L^p(G)$, it follows from Lemma 5(b) that N_p has a compact unit ball and hence is finite dimensional.

Since N_p is of finite dimension, it has a closed complement L in $L^p(G)$, i.e. every u in $L^p(G)$ can be written uniquely in the form $u = w + v$, where w lies in N_p and v lies in L. If u is an element of $D(A_p)$, so is v, since each w lies in $D(A_p)$. It follows that if we restrict A_p to $D(A_p) \cap L$, the range of A_p and the range of this restricted mapping A_p^+ are the same. On the other hand A_p^+ is a closed mapping since L is a closed subspace of $L^p(G)$ by construction.

Since A_p^+ is a closed mapping, the assertion that $R(A_p) = R(A_p^+)$ is closed is equivalent to the continuous invertibility of the one-to-one mapping A_p^+ on its range. Suppose that A_p^+ is not invertible. Then there exists a sequence $\{u_k\}$ from $D(A_p) \cap L$ such that $\|A_p u_k\|_{L^p} \to 0$, while $\|u_k\|_{L^p} = 1$ for all k. By Theorem 2, the sequence $\{u_k\}$ is bounded in $W^{2m,p}(G)$, so that if we replace it by a subsequence, we may assume that u_k converges weakly in $W^{2m,p}(G)$ to an element u_0 of the latter space. But A_p is a continuous mapping of $W^{2m,p}(G)$ into $L^p(G)$, hence weakly continuous, so that $A u_k \to A u_0$ weakly in $L^p(G)$. But $A u_k \to 0$ strongly in $L^p(G)$, so that $A u_0 = 0$. Thus u_0 lies in N_p. On the other hand, L is a closed linear subspace of $L^p(G)$, hence weakly closed, and u_0, being the weak limit of elements u_k of L, must lie in L. Finally the weak convergence of u_k to u_0 in $W^{2m,p}(G)$ implies by Lemma 5(b) that u_k converges strongly to u_0 in $L^p(G)$, (since the imbedding mapping of $W^{2m,p}(G)$ in $L^p(G)$ for bounded G is a compact map). In particular, $\|u_0\|_{L^p} = \lim_k \|u_k\|_{L^p} = 1$. Thus we have a non-zero element u_0 lying in $L \cap N$, while by assumption $L \cap N = \{0\}$. This contradicts the assumption that $R(A_p)$ is not closed, so that $R(A_p)$ must be closed in $L^p(G)$.

We now complete Theorem 9 with the following result in which the regularity assumptions of Theorem 9 are replaced by assumptions on the spectrum of the homogeneous operator on bounded domains.

Theorem 12. *Let G be an open subset of E^n which is uniformly regular of class C^{2m} and locally regular of class C^{4m}. Let A be a regularly elliptic operator of order $2m$ on G with coefficients satisfying the regularity condition* (IV), *A' the formal adjoint of A. Let $1 < p < \infty$, $q = p(p-1)^{-1}$. Suppose that for each bounded sub-domain G' of G which is uniformly regular of class C^{4m}, the realization of A^0, the top-order homogeneous part of A, under null Dirichlet boundary conditions of order m in $L^2(G')$, has at least one point in its resolvent set. Let Γ_0 be a subset of the boundary Γ of G satisfying the regularity condition of Lemma* 10, *$A_{p,0}$, $A_{p,1}$, $A'_{q,0}$, and $A'_{q,1}$ the realizations of A and A' in $L^p(G)$ and $L^q(G)$ respectively defined with respect to Γ_0 in Section* 2.

Then $(A_{p,0})^* = A'_{q,1}$, $(A_{p,1})^* = A'_{q,0}$, and $A_p^* = A'_p$.

Proof of Theorem 12. The proof is identical with that of Theorem 9 up to the point following equation (3.28), where one has a function v in $D(A'_{2,1})$ for which $A'v = g$ with v and g in $L^q(G)$, $q \geqq 2$, and one wishes to show that v lies in $D(A'_{q,1})$. If λ is any function from $C_c^{2m}(G \cup \Gamma_0)$, the new function $w = \lambda v$ lies in $W^{2m,2}(G)$, and indeed in $D(A'_2)$, and satisfies the equation $A'w = \lambda g + \sum_{|\beta| < 2m} c_\beta(x) D^\beta v$. Since v lies in $W^{2m,2}_{\mathrm{loc}}(G \cup \Gamma_0)$ while the functions $c_\beta(x)$ have compact support in $G \cup \Gamma_0$, the last summand in the expression for $A'w$ lies in $L^r(G)$, where $2 \leqq r \leqq 2n(n-p)^{-1}$. If g lies in $L^q(G)$, and q is in the range of r above, then $A'w$ lies in $L^q(G)$ and from Theorem 10, we find that w lies in $W^{2m,q}(G)$. If $q > 2n(n-2)^{-1}$, we remark that w lies in $L^r(G)$ and $A'w$ lies in $L^r(G)$, (since G is bounded in the special case to which the argument is reduced) for $r = 2n(n-2)^{-1}$. Applying Theorem 10, we find that w lies in

$W^{2m,r}(G)$, and hence u lies in $W^{2m,r}_{\text{loc}}(G \cup \Gamma_0)$. But then $A'w$ lies in an L^r class for higher r, and applying the obvious boot-strap procedure, we reach the conclusion of the Theorem.

Using Theorem 12, we can establish a useful extension of Theorem 10.

Theorem 13. *Let A be a regularly elliptic operator of order $2m$ on an open set G of E^n which is uniformly elliptic of class C^{2m} and locally regular of class C^{4m}. Suppose that A satisfies the regularity conditions* (I) *and* (II), *and that for each bounded sub-domain G' of G, A satisfies condition* (IV) *with respect to G'. Suppose further that for every bounded open subset G' of G which is uniformly regular of class C^{4m} that $A^0_{2,(G')}$ has a non-empty resolvent set, where $A^0_{2,(G')}$ is the realization of the top order homogeneous part of A under null Dirichlet boundary conditions of order m in $L^2(G')$. Let $1 < p \leqq p_1 < \infty$. Then:*

(a) *If $A_{p,(G')}$ is the realization of A in $L^p(G')$ under null Dirichlet boundary conditions of order m, for G' as above, then the spectrum of $A_{p,(G')}$ is identical with that of $A_{2,(G')}$. In particular, $(A_2 - \zeta I)^{-1}$ exists for a bounded set G, and a complex number ζ if and only if $(A_p - \zeta I)^{-1}$ exists for that G and ζ.*

(b) *If u lies in $D(A_p)$ with Au in $L^{p_1}(G)$, then $u \in D(A_{p_1})$.*

Proof of Theorem 13. A careful examination of the proof of Theorem 10 shows that the only use which we made of the assumption that $(A^0_2 - \zeta I)^{-1}$ exists on a bounded sub-domain G' lay in the following argument: We are given u in $D(A_p)$ with $A^0 u \in L^{p_1}(G)$, $p \leqq p_1 \leqq np(n-p)^{-1}$, we know that $(A^0_p)^{-1}$ is a bounded linear mapping defined on all of $L^p(G'')$, we approximate A^0 by differential operators $A^{(k)}$ with analytic coefficients so that for k sufficiently large, $A^{(k)}_p$ exists and for v in $L^p(G')$, $(A^{(k)}_p)^{-1}v$ converges to $A_p^{-1}v$ in $W^{2m,p}(G')$. For the operators $A^{(k)}_p$, we know by Theorem 8 and Theorem 2 that $\|(A^{(k)}_p)^{-1}v\|_{W^{2m,p_1}(G)}$ is uniformly bounded for v in $L^{p_1}(G')$, and therefore by a uniqueness of limits argument, the limit of $(A^{(k)}_p)^{-1}$ in $L^2(G')$, i.e. $(A^0_p)^{-1}v$, must lie in $W^{2m,p_1}(G)$. Choosing $v = A^0 u$, the conclusion follows. In Theorem 10, we apply a step-by-step argument starting with $p = 2$, and establish conclusion (b) of Theorem 13 for all p with $p \geqq 2$. But then conclusion (a) of Theorem 13 follows easily for $p \geqq 2$, since the null space of $(A_p - \zeta I)$ on a bounded set G' lies in the null space of $(A_2 - \zeta I)$ by Hölder's inequality, while the null space of $(A_2 - \zeta I)$ lies in the null space of $(A_p - \zeta I)$ by conclusion (b) which we have already established. Similarly if $f \in R(G)$, f lies in the range of $(A_2 - \zeta I)$ if and only if it lies in the range of $(A_p - \zeta I)$ by (b), and since both ranges are closed in their respective image spaces, (a) follows for $p \geqq 2$.

But if (a) is true for $p \geqq 2$, it must be true under the hypotheses of Theorem 13 for $1 < p \leqq 2$. Indeed A' satisfies the same conditions as A and has the same top-order part A^0. Moreover, $(A'_2 - \bar{\zeta} I) = (A_2 - \zeta I)^*$. If $(A_2 - \zeta I)^{-1}$ exists, so does $(A'_2 - \bar{\zeta} I)^{-1}$, and by our preceding argument applied to A', so does $(A'_p - \bar{\zeta} I)^{-1}$ for $p \geqq 2$. But by Theorem 12 applied to the bounded domain G', if $q = p(p-1)^{-1}$, we have $(A_q - \zeta I) = (A'_p - \bar{\zeta} I)^*$, and hence $(A_q - \zeta I)^{-1}$ exists for all q with $1 < q \leqq 2$. The converse follows by a parallel argument, and (a) is established for all exponents p with $1 < p < \infty$.

Applying (a) to the homogeneous operator A^0, however, we see that $A_p^0 - \zeta I$ is invertible for all $p > 1$, under our hypotheses, and by our previous remarks on the proof of Theorem 10, conclusion (b) follows for all p and p_1 with $1 < p \leqq p_1 < \infty$.

While in Theorems 12 and 13, we obtain the same conclusions as in Theorem 9, with the assumptions (V) and (U) in Theorem 9 replaced by the assumption that the Dirichlet problem for A in $L^2(G')$, for each bounded regular subdomain G' of G, has at least one point in its resolvent set (an assumption which is verified for all uniformly strongly elliptic A in Section 4), it is of interest to obtain the conclusions of Theorem 9 under the regularity condition (V) for any regularly elliptic operator A without the use of the incompletely verifiable assumption (U) on the uniqueness of the Cauchy problem in the small. In the following theorem, we shall show using a variant of the proof of Theorem 12 that the conclusions of Theorem 9 are valid without assuming (U).

Theorem 14. *Let A be a regularly elliptic differential operator of order $2m$ on an open subset G of E^n, with G uniformly regular of class C^{2m} and locally regular of class C^{4m}. Suppose that A satisfies the regularity condition* (IV) *of Section* 2 *as well as* (V) *of Section* 3. *Then the conclusions of Theorem* 9 *are valid. Moreover, if $1 < p \leqq p_1 < \infty$, and $u \in D(A_p)$ with $Au \in L^{p_1}(G)$, then u lies in $D(A_{p_1})$.*

Proof of Theorem 14. The second conclusion of Theorem 14 follows from the first by a boot-strap argument, for it suffices to consider G bounded and in the range $p \leqq p_1 \leqq np(n-2mp)^{-1}$ the fact that u lies in $W^{2m,p}(G)$ implies that u lies in $L^{p_1}(G)$. Thus for all f in $R(G) \cap D(A'_{q_1})$, $q_1 = p_1(p_1-1)^{-1}$, we have $(u, A'f) = (Au, f)$, where Au lies in $L^{p_1}(G)$. Thus u lies in $D((A'_{q_1})^*)$, and by the conclusions of Theorem 9, u must lie in $D(A_{p_1})$. Hence, it suffices to prove the conclusions of Theorem 9 under our present hypotheses.

Following the pattern of the proofs of Theorems 9 and 12, we may assume that G is a bounded domain which is uniformly regular of class C^{4m}, that we are given $p \geqq 2$ with $(A_p)^* = A'_q$ and a function v in $D(A_p)$ such that Av lies in $L^{p_1}(G)$ with p_1 satisfying the inequality $p \leqq p_1 \leqq np(n-p)^{-1}$, and it suffices to show that v lies in $W^{2m,p_1}(G)$. By Lemma 5, we know that v itself lies in $L^{p_1}(G)$.

Let $R(A_p)$ be the range of A_p in $L^p(G)$. By Theorem 11, since G is bounded, $R(A_p)$ is a closed subspace of $L^p(G)$. Its annihilator in $L^q(G)$ is precisely the null space of A'_q, since our assumptions include the fact that $(A_p)^* = A'_q$, and hence $R(A_p)$ is a subspace of finite co-dimension in $L^p(G)$, again by Theorem 11. Let $\{v_1, \ldots, v_s\}$ be a linearly independent base for the null space of A'_q in $L^q(G)$. Each v_k is a solution of the equation $A'v_k = 0$, and if we apply Theorem 8 to the operator A^h which is the top-order part of A' and, write $A' = A^h + S$ with S of lower order, then v_k is a solution for each k of the equation $A^h v_k = -Sv_k$ and a boot-strap argument yields the fact that each v_k lies in $W^{2m,r}(G)$ for every r, $1 < r < \infty$. In particular, $v_k \in D(A'_2) \cap D(A_p)$, and choosing a new basis, we may assume them orthogonal in $L^2(G)$. Let P be the mapping

of $L^p(G)$ into $R(A_p)$ defined by $Pu = u - \sum_k (u, v_k) v_k$, where we assume that $(v_k, v_{k_1}) = \delta_{k,k_1}$. Then P is an idempotent map of $L^p(G)$ onto $R(A_p)$. Similarly, let $\{w_1, \ldots, w_t\}$ be a linearly independent basis for the (finite-dimensional) null-space of A_p. By the same argument, each w_k lies in $W^{2m,r}(G)$ for every $r > 1$. We define $Qu = u - \sum_k (u, w) w_k$ as a mapping of $L^p(G)$ onto the annihilator L of the null-space of A, where again we assume the basis $\{w_k\}$ chosen so that $(w_k, w_{k_1}) = \delta_{k,k_1}$. Then Q is bounded idempotent mapping of $L^p(G)$ into itself and maps $D(A_p)$ into itself. Moreover, since $A w_k = 0$ for each k, we have $A Q u = A u$ for all u in $D(A_p)$, and the range of $A_p Q$ equals the range of A_p. In addition, $A_p Q$ is a one-to-one mapping of $L \cap D(A_p)$ onto $R(A_p)$, and since the latter subspace is closed in $L^p(G)$, it follows from the closed graph theorem that the restriction of A_p to $L \cap D(A_p)$ has a bounded inverse defined on $R(A_p)$. By Theorem 2, this inverse mapping, which we shall denote by J, is not only a bounded mapping of $R(A_p)$ into $L^p(G)$, but also a bounded linear map of $R(A_p)$ into $W^{2m,p}(G)$.

Let $\{j_k(x)\}$ be a sequence of functions from $C_c^\infty(E^n)$ converging to the Dirac delta function in our usual sense, and for each α with $|\alpha| \leq 2m$, let $a_{\alpha,k}(x) = \int_G j_k(x - y)\, a_\alpha(y)\, dy$. The function $a_{\alpha,k}$ is infinitely differentiable on E^n and may be uniformly approximated by an analytic function, which we shall also designate by $a_{\alpha,k}$. If we define $A^{(k)}$ to be $A^{(k)} = \sum_{|\alpha| \leq 2m} a_{\alpha,k}(x) D^\alpha$, it follows from our construction that for all α, $a_{\alpha,k}$ will converge uniformly to a_α as $k \to \infty$. If we consider $A_p^{(k)}$ restricted to $L \cap D(A_p)$, these mappings will converge in norm to A_p as k becomes large. We let $T^{(k)} = P A_p^{(k)}$, where each $T^{(k)}$ is defined on $L \cap D(A_p)$. The map $T^{(k)}$ maps $L \cap D(A_p)$ into $R(A_p)$ for every k, and for every u,

$$T^{(k)} u = A_p^{(k)} u - \sum_j (A_p^{(k)} u, v_j) v_j .$$

Thus,

$$\|T^{(k)} u - A_p u\|_{L^p(G)} \leq \|A_p^{(k)} u - A_p u\|_{L^p(G)} + \sum_j |(A_p^{(k)} u, v_j)|\, \|v_j\|_{L^p(G)} .$$

Since $(A_p u, v_j) = 0$ for every j, we have $|(A_p^{(k)} u, v_j)| \leq \|(A_p^{(k)} u - A_p u\|_{L^p(G)} \times \times \|v_j\|_{L^q(G)}$. Combining these inequalities, we have

$$\|T^{(k)} u - A_p u\|_{L^p(G)} \leq K \|u\|_{W^{2m,p}(G)} \cdot \varepsilon(k) ,$$

where $\varepsilon(k) \to 0$ as $k \to \infty$. Since A_p considered as a mapping from a subset of $W^{2m,p}(G)$ into $R(A_p)$, considered as a subset of $L^p(G)$, has a bounded inverse, for sufficiently large k, $T^{(k)}$ will have a bounded inverse $J^{(k)}$ mapping $R(A_p)$ into $W^{2m,p}(G)$, while for each v in $R(A_p)$, $J^{(k)} u \to J u$ in $W^{2m,p}(G)$ and, by Lemma 5, also in $L^{p_1}(G)$. On the other hand, $T^{(k)} J^{(k)} u = u$, i.e.

$$A_p^{(k)} J^{(k)} u = u + \sum_j (A_p^{(k)} J^{(k)} u, v_j) v_j .$$

Let $u = A_p v$ for our given function v in $D(A_p)$. Then $J u = v - \sum_j (v, w_j) w_j$, while $A_p^{(k)} J^{(k)} u$ lies in $L^{p_1}(G)$ by hypothesis on $A_p v$ and by our previous

remarks concerning the v_j. But the operator $A^{(k)}$ has analytic coefficients, so that by Theorem 10, $J^{(k)}u$ lies in $W^{2m,p_1}(G)$. By Theorem 2, since the operators $A^{(k)}$ are all uniformly elliptic with a fixed constant of ellipticity for large k and all bounds are uniform,

$$\|J^{(k)}u\|_{W^{2m,p_1}(G)} \leqq K\{\|A_p v\|_{L^{p_1}(G)} + \|A_p v\|_{L^p(G)} + \|J^{(k)}u\|_{L^{p_1}}\}.$$

Hence the sequence $\{J_u^k\}$ being bounded in $W^{2m,p_1}(G)$, we may extract a subsequence (which we can identify with the original sequence) which converges weakly in $W^{2m,p_1}(G)$ to an element v_1 in $W^{2m,p_1}(G)$. But $J^{(k)}u$ converges strongly to $Ju = v - \sum_j (v, w_j)w_j$, in $L^p(G)$, so that by the uniqueness of limits

$$v = v_1 + \sum_j (v, w_j)w_j,$$

v itself must lie in $W^{2m,p_1}(G)$, and our proof is complete.

Section 4: Strongly elliptic operators

We turn in the present section to a detailed consideration of the spectrum of A_p for A uniformly strongly elliptic.

Theorem 15. *Let A be a regularly elliptic operator on an open subset G of E^n, with G uniformly regular of class C^{2m}. Suppose that A is uniformly strongly elliptic on G and satisfies the regularity conditions* (I) *and* (II) *of Section* 1. *Then:*

(a) *Given $\varepsilon > 0$, there exists $K(\varepsilon) > 0$, such that for all real $\xi \geqq 0$ and all u in $D(A_2)$, we have for all $1 > \varepsilon > 0$.*

$$\|(A_2 + \xi I)u\|_{L^2(G)} \geqq \{(1-\varepsilon)\xi - K(\varepsilon)\}\|u\|_{L^2(G)}. \tag{4.1}$$

(b) *If the top-order coefficients of A, a_α for $|\alpha| = 2m$, satisfy a Hölder condition of order λ_1, $0 < \lambda_1 \leqq 1$, then the function $K(\varepsilon)$ of the inequality* (4.1) *of* (a) *may be chosen to be $K(\varepsilon) = K_0\varepsilon^{-\mu}$, where $\mu = m(\lambda_1 + 1)\lambda_1^{-1}$, (and in particular depends only on m and λ_1).*

(c) *Under the hypotheses of* (a), *if G is a bounded set, then the resolvent set of A_p for any p with $1 < p < \infty$ is the same for all p and contains all but a bounded part of every ray in the complex plane emanating from the origin and making an angle of less then $\pi/2$ with the negative real axis.*

(d) *Suppose that the top-order coefficients of A satisfy a Hölder condition of order λ on G and that G is bounded. Then there exists a constant $K_1 > 0$ such that the set*

$$\{\zeta : \zeta \text{ a complex number, } \mathrm{Re}(\zeta) \leqq -K_1(1 + |I_m(\zeta)|^{2\lambda(\lambda+1)^{-1}})\}$$

is contained in the resolvent set of A_p for all p with $1 < p < \infty$, while for real $\xi \geqq \xi_0$, we have,

$$\|(A_2 + \xi I)^{-1}\| \leqq (\xi - K_1\xi^\lambda - K_1)^{-1},$$

where $\lambda = \mu(1+\mu)^{-1}$, μ is the constant defined in (b), *and $0 < \lambda < 1$.*

Theorem 16. *Let A be a regularly elliptic operator on an open subset G of E^n with G uniformly regular of class C^{2m} and locally regular of class C^{4m}. Suppose*

that A is uniformly strongly elliptic on G and satisfies the regularity condition (IV) *of Section* 3. *Then:*

(a) *For $1 < p < \infty$, $q = p(p-1)^{-1}$, A' the formal adjoint of A, we have $(A_p)^* = A'_q$.*

More generally, if $A_{p,0}$, $A_{p,1}$, $A'_{q,0}$, and $A'_{q,1}$ are the realizations of A and A', respectively defined as in Section 2 *with respect to a portion Γ_0 of the boundary Γ of G which satisfies the regularity condition of Lemma* 10, *then*

$$(A_{p,0})^* = A'_{q,1},\ (A_{p,1})^* = A'_{q,0}\,.$$

(b) *There exists a constant $K_2 > 0$ (depending only upon the domain G, the constant of uniform ellipticity, the bounds on the coefficients a_α and a'_α, and the modulus of continuity of the top order coefficients of A, all on the domain G) such that the set*

$$\{\zeta : \mathrm{Re}(\zeta) \leqq -K_2\}$$

is contained in the resolvent set of A_2, while for real $\xi \geqq \xi_0$, we have the inequality

$$\|(A_2 + \xi I)^{-1}\| \leqq (\xi - K_2)^{-1}.$$

(c) *There exists a constants $c > 0$, $K_3 \geqq 0$ such that for u in $D(A_2)$, $\mathrm{Re}(Au, u) \geqq c\|u\|^2_{m,2} - K_3\|u\|^2_{0,2}$.*

Theorem 17. *Let A be a regularly elliptic operator on an open subset G of E^n, with G uniformly regular of class C^{2m}. Suppose that A is uniformly strongly elliptic on G and satisfies the regularity conditions* (I) *and* (II). *Suppose G is bounded. Then for any p with $1 < p < \infty$, the transformation $(A_p - \xi I)^{-1}$ in $L^p(G)$, which exists for ξ large and negative by Theorem* 15*(c), is a compact linear transformation in $L^p(G)$, the spectrum of A_p is discrete, the eigenspaces are of finite dimension, and $(A_p - \zeta I)^{-1}$ will exist for any complex ζ for which $(A_p - \zeta I)$ has a trivial null-space.*

Proof of Theorem 15. Let $u \in D(A_2)$. Then for ξ real,

$$\|(A_2 + \xi I)u\|^2_{L^2} = (Au, Au) + \xi^2(u \cdot u) + 2\xi\,\mathrm{Re}(Au, u)\,. \tag{4.1}$$

By Theorem 2, $(Au, Au) = \|Au\|^2_{L^2} \geqq c_0\|u\|^2_{2m,2} - K\|u\|^2_{0,2}$, with $c_0 > 0$, $K \geqq 0$. Hence, (4.1) implies that

$$\|(A_2 + \xi I)u\|^2_{0,2} \geqq c_0\|u\|^2_{2m,2} - K\|u\|^2_{0,2} + 2\xi\,\mathrm{Re}(Au, u) + \xi^2\|u\|^2_{0,2}. \tag{4.2}$$

(For the remainder of this section, we shall use the notation $\|u\|$ for the L^2-norm of u, and $\|u\|_j$ for the $W^{j,2}$-norm of u.)

Suppose $\delta > 0$ is given. Let $\{\eta_k(x)\}$ be a family of functions from $C^{2m}_c(E^n)$ having the properties described in Lemma 2 corresponding to the given δ, and in particular having supports of diameter less than δ. We write A as the sum of its top-order and lower order terms, $A = A^h + S$, A^h homogeneous of order $2m$ and S of order $< 2m$. Then

$$(Au, u) = (A^h u, u) + (Su, u) \tag{4.3}$$

while

$$(A^h u, u) = \sum_{k=1} (A^h(\eta_k^2 u), u) = \sum_k (A^h(\eta_k u), \eta_k u) + R_1(u)\,, \tag{4.4}$$

where $R_1(u)$ being a sum of the form

$$R_1(u) = \sum_k \sum_{|\alpha|=2m,\,|\beta|<2m} c_{\alpha,\beta}(a_\alpha D^{\alpha-\beta}(\eta_k) D^\beta(\eta_k u), u)\,, \tag{4.5}$$

satisfies an estimate of the form

$$|R_1(u)| \leqq K\|u\| \left\{ \sum_{0\leqq r<2m} \delta^{r-2m}\|u\|_r \right\}. \tag{4.6}$$

(The boundedness of the summation on k for fixed α and β in (4.5) follows from (c) of Lemma 5.)

Let $w(\delta) = \sum_{|\alpha|=2m} \sup_{x,y\in G,\,|x-y|<\delta} \{|a_\alpha(x) - a_\alpha(y)|\}$. By the hypothesis of Theorem 15(a), $w(\delta) \to 0$ as $\delta \to 0$. Furthermore, if $B_k = \sum_{|\alpha|=2m} a_\alpha(x_k)\, D^\alpha$ with x_k a point of the set N_k of the covering of Lemma 2, we have

$$(A^h(\eta_k u), \eta_k u) = (B_k(\eta_k u), \eta_k u) + R_{2,k}(u)\,, \tag{4.7}$$

where

$$\sum_k |R_{2,k}(u)| \leqq K w(\delta)\, \|u\|_{2m} \cdot \|u\|\,. \tag{4.8}$$

By Lemma 9, the function u which lies in $D(A_2)$ is the limit in $W^{m,2}(G)$ of functions from $C_c^\infty(G)$. It follows that we may apply the van Hove-Garding inequality ([68], [29]) to the strongly elliptic operator B_k with constant coefficients and to the function $\eta_k u$ for each k. We obtain

$$\mathrm{Re}(B_k(\eta_k u), \eta_k u) \geqq 0\,. \tag{4.9}$$

(The choice of $i^{-1}\,\partial/\partial x_j$ and not $\partial/\partial x_j$ as our elementary differential operator causes the disappearance of the factor $(-1)^m$ which usually appears in inequalities of the type of (4.9). It is this choice that causes all our operators A_p to have their spectrum in the right half of the complex ζ-plane.)

Finally, it follows from the Schwarz inequality that $|(Su, u)| \leqq K\|u\|_{2m-1}\|u\|$. Combining (4.3), (4.4), (4.6), (4.7), (4.8), and (4.9) with this last remark, we obtain

$$\mathrm{Re}(Au, u) \geqq -K\|u\| \cdot \left\{ \sum_{0\leqq r\leqq 2m-1} \delta^{r-2m}\|u\|_r + w(\delta)\|u\|_{2m} \right\}. \tag{4.10}$$

Introducting (4.10) into (4.2), we obtain

$$\|(A_2 + \xi I)u\|^2 \geqq c_0\|u\|_{2m}^2 + (\xi^2 - K)\|u\|^2 - K\xi\|u\| \times \left\{ \sum_{r\leqq 2m} \delta^{r-2m}\|u\|_r + w(\delta)\|u\|_{2m} \right\}. \tag{4.11}$$

Let $\varepsilon > 0$ be given, as in the hypothesis of Theorem 15. By the inequality between the geometric and arithmetic means,

$$K\xi\|u\| \cdot w(\delta)\|u\|_{2m} \leqq \frac{\varepsilon\xi^2}{2}\|u\|^2 + K^2(2\varepsilon)^{-1} w(\delta)^2\|u\|_{2m}^2\,. \tag{4.12}$$

We now fix δ in terms of ε, where δ is made so small that

$$2K^2 w(\delta)^2 \leqq c_0\varepsilon\,. \tag{4.13}$$

Then the coefficient of $\|u\|^2_{2m}$ on the right side of (4.12) is less than $4^{-1}c_0$. On the other hand, δ now being fixed,

$$K\xi \cdot \|u\| \cdot \delta^{r-2m}\|u\|_r \leqq \varepsilon(4m)^{-1}\xi^2\|u\|^2 + m\varepsilon^{-1}K^2 + \delta^{2r-4m}\|u\|^2_r . \tag{4.14}$$

By Lemma 6, however, $\|u\|^{2m}_r \leqq K\|u\|^r_{2m} \cdot \|u\|^{2m-r}$. Applying the familiar inequality $ab \leqq p^{-1}a^p + q^{-1}b^q$, for $p^{-1}+q^{-1}=1$, $a, b \geqq 0$, with $p = 2mr^{-1}$, $q = 2m(2m-r)^{-1}$, we obtain

$$m\varepsilon^{-1}K^2\delta^{2r-4m}\|u\|^2_r \leqq c_0(4m)^{-1}\|u\|^2_{2m} + ((4m)^r c_0^{-r} m^{2m} K^{4m})^{(2m-r)^{-1}} \varepsilon^{-2m(2m-r)^{-1}}\delta^{-4m}\|u\|^2 . \tag{4.15}$$

Combining (4.12), (4.)13, (4.14), and (4.15) with (4.11), we obtain

$$\|(A_2 + \xi I)u\|^2 \geqq c_0/4\,\|u\|^2_{2m} + \{(1-\varepsilon)\xi^2 - K_1\varepsilon^{-2}\delta^{-4m}\}\|u\|^2, \tag{4.16}$$

where δ is related to ε by (4.13). Dropping the first term on the right hand side which is non-negative, taking square roots of both sides, and using the fact that $(a^2-b^2)^{1/2} \geqq a-b$ and that $(1-\varepsilon)^{1/2} \geqq 1-\varepsilon$, we obtain

$$\|(A_2 + \xi I)u\| \geqq \{(1-\varepsilon)\xi - K_2\varepsilon^{-m}\delta^{-2}\}\|u\| . \tag{4.17}$$

Theorem 15(a) is obtained by setting $K(\varepsilon) = K_2\varepsilon^{-m}\delta^{-2m}$, for some δ such that (4.13) holds.

Theorem 15(b) is obtained from (4.17) and the fact that if the top order coefficients of A satisfy a Hölder condition with exponent λ_1, then the function $w(\delta) \leqq K\delta^{\lambda_1}$. Thus the δ for which (4.13) is satisfied may be taken as $\delta = K\varepsilon^{(2\lambda_1)^{-1}}$. Setting this value of δ in (4.17), we see that

$$K(\varepsilon) = K_0\varepsilon^{-m(1+\lambda_1)\lambda_1^{-1}}, \tag{4.18}$$

where the exponent is the negative of the constant μ defined in (b). Thus the assertion of (b) has been established.

Suppose now that G is a bounded set in E^n. We may approximate A by a sequence of differential operators $\{A^{(k)}\}$ with infinitely differentiable coefficients on E^n with the top-order coefficients of the $A^{(k)}$ converging uniformly on G to those of A and the lower order coefficients converging boundedly to those of A a.e. in G. In particular, the inequalities of Theorem 2 and the inequality (4.1) will hold for all the $A^{(k)}$ with a uniform constant $K(\varepsilon)$ for each fixed $\varepsilon > 0$. For operators $A^{(k)}$ with infinitely differentiable coefficients, it follows from the variational theory of the Dirichlet problem ([3], [5], [31], [69] for the existence theory, [6] and [52] for the proof of assumption of boundary values) that for any complex number ζ for which the equation $(A_2^{(k)} - \zeta I)u = 0$ has no non-trivial solutions, $(A_2^{(k)} - \zeta I)^{-1}$ exists as a bounded operator on all of $L^2(G)$, i.e. $A_2^{(k)} - \zeta I$ maps onto $L^2(G)$. Suppose, therefore, that for some fixed value of ζ and all k, we have an inequality of the form $\|(A_2^{(k)} - \zeta I)u\| \geqq K_0\|u\|$, with $K_0 > 0$, for all u in $D(A_2)$. Then for each k, $(A_2^{(k)} - \zeta I)^{-1}$ will exist on all of $L^2(G)$, and we will have the inequality $\|(A_2^{(k)} - \zeta I)^{-1}\| \leqq K_0^{-1}$ for all k. We may assume that $\zeta = 0$. Let v be an arbitrary element of $L^2(G)$, u_k the unique solution of the equation $(A_2^k)\,u_k$. We know that $\|u_k\| \leqq K_0^{-1}\|v\|$ for all k, and by Theorem 2,

with a constant uniform in k, $\|u_k\|_{2m} \leqq K_1\|v\|$. Extracting a subsequence if necessary, we may assume that u_k converges weakly to u_0 in $W^{2m,2}(G)$. Since $D(A_2)$ is closed and hence weakly closed in $W^{2m,2}(G)$, u_0 lies in $D(A_2)$. Since G is bounded, it follows from Lemma 5 that the sequence $\{u_k\}$ converges strongly in $W^{2m-1,2}(G)$. If $A^{(k)} = A^{(k),h} + S^{(k)}$, $A^{(k),h}$ top-order homogeneous and $S^{(k)}$ of lower order, we have $A^{(k)}u_k = A^{(k),h}u_k + S^{(k)}u_k$. Since $A^{(k),h}$ converges in norm to A^h as a mapping from $W^{2m,2}(G)$ into $L^2(G)$, it follows from the weak convergence of the u_k to u_0 that $A^{(k),h}u_k$ converges weakly to $A^h u_0$ in $L^2(G)$. Since u_k converges strongly to u_0 in $W^{2m-1,2}(G)$ and since the coefficients of $S^{(k)}$ converge boundedly a.e. in G to those of S, we have

$$\|S^{(k)}u_k - Su\| \leqq \|(S^{(k)}u_k - S^{(k)}u\| + \|(S^{(k)} - S)u\| \leqq K\|u_k - u\|_{2m-1} + \\ + \sum_{|\beta| \leqq 2m-1} \|(a_{\alpha,k} - a_\alpha)u\| .$$

The first term on the right goes to zero by the strong convergence of the u_k to u_0 in $W^{2m-1,2}(G)$, the second because of dominated convergence. Thus, $S^{(k)}u_k$ converges strongly to Su_0 in $L^2(G)$. It follows that $A^{(k)}u_k$ converges weakly to Au_0 in $L^2(G)$ as $k \to \infty$, but since $A^{(k)}u_k = v$ for all k, it follows that $Au_0 = v$.

In particular, the inequality (4.1) of Theorem 15(a) implies that for ξ large and negative, $(A_2 - \xi I)$ will have a trivial null-space, and this will be true uniformly for large k for any approximating sequence $\{A^{(k)}\}$. Hence $(A_2 - \xi I)^{-1}$ will exist for ξ large and negative. It follows from Theorem 2 that for any ζ for which $(A - \zeta I)^{-1}$ exists, that it is a bounded mapping from $L^2(G)$ into $W^{2m,2}(G)$. Hence by Lemma 5, $(A_2 - \zeta I)^{-1}$ is a compact linear operator in $L^2(G)$ for any ζ for which it exists. Since it exists for at least one value of ζ and indeed for all ζ which are large and negative, it follows that $(A_2 - \zeta I)^{-1}$ will exist for any ζ for which $(A_2 - \zeta I)$ has a trivial null space. Conversely, if $(A_2 - \zeta I)$ has a non-trivial null space, then $(A_2 - \zeta I)$ has a range which is a proper subset of $L^2(G)$. Thus, the discussion of the spectrum of A_2 on a bounded domain is reduced to the discussion of point eigenvalues.

Let $\zeta = \xi + i\eta$, ξ and η real. Then,

$$(A_2 - \zeta I) = (I + (\xi_0 - \zeta)(A_2 - \xi_0 I)^{-1})(A_2 - \xi_0 I) , \tag{4.19}$$

for any ξ_0 for which $(A_2 - \xi_0 I)^{-1}$ exists on all on $L^2(G)$. Thus $(A_2 - \zeta I)^{-1}$ will exist provided that the bounded operator $I + (\xi_0 - \zeta)(A_2 - \xi_0 I)^{-1}$ has an inverse defined on all of $L^2(G)$ for some ξ_0 in the resolvent set of A_2. This will certainly be the case if

$$|\zeta - \xi_0| < \{\|(A_2 - \xi_0 I)^{-1}\|\}^{-1} \tag{4.20}$$

for some such ξ_0. Taking ξ_0 real, large, and negative, and using the estimate (4.1) which we have already established, we see that (4.20) will hold if

$$|\zeta - \xi_0| < (1 - \varepsilon)|\xi_0| - K(\varepsilon) , \quad (K(\varepsilon) \text{ as in Theorem 15(a)}) \tag{4.21}$$

for any ε between 0 and 1. Suppose that ζ is taken on a ray in the left half ζ-plane emanating from the origin, i.e. $\eta = c\xi$, c real and ξ negative, (c held fixed, ξ variable with $\xi \leqq 0$). Performing an elementary calculation with (4.21), Theorem 15(c) follows.

To establish Theorem 15(d), finally, it suffices by an argument like the one of the preceeding paragraph to establish an inequality of the form

$$\|(A_2 + \xi I)u\| \geqq (\xi - K_1\xi^{\lambda} - K_1)\|u\| \tag{4.22}$$

for all u in $D(A_2)$. By Theorem 15(b), which we have already proved, we have for $\xi \geqq 0$

$$\|(A_2 + \xi I)u\| \geqq (1-\varepsilon) - K_0\varepsilon^{-\mu} \tag{4.23}$$

for any ε with $0 < \varepsilon < 1$. We set $\varepsilon = (\xi + 1)^{-(1+\mu)^{-1}}$, and (4.23) becomes

$$\|(A_2 + \xi I)u\| \geqq \{\xi - K_1\xi\lambda - K_1\}\|u\|, \tag{4.24}$$

with $\lambda = \mu(1+\mu)^{-1}$, as demanded in (4.22). Thus (4.22) has been established, and Theorem 15(d) is proved. To finish the proof of Theorem 15, we must consider $p \neq 2$ in Theorem 15(c).

By Theorem 10, however, the null space of $(A_p - \zeta I)$ with ζ fixed is the same for all p, $1 < p < \infty$, for since G is bounded, all functions in $W^{2m,p}(G)$, lie in $W^{2m,r}(G)$ for $r < p$. On the other hand, if ζ lies in the resolvent set of A_2 and f is an element of $R(G)$, then the solution u of $(A_2 - \zeta I)u = f$ must lie in $D(A_p)$ for every p by Theorem 10. Hence, for ζ in the resolvent set of A_2, $(A_p - \zeta I)$ is one-to-one, its range is closed by Theorem 11 since G is a bounded set in E^n, and its range contains the dense subset $R(G)$ of $L^p(G)$ by the remark of the last sentence. Hence $(A_p - \zeta I)^{-1}$ exists on all of $L^p(G)$ and is a bounded linear operator on $L^p(G)$. The converse conclusion that if ζ lies in the resolvent set of A_p, then it must lie in the resolvent set of A_2, holds by an identical argument. Thus the resolvent sets of all the operators A_p are the same, and the proof of Theorem 15(c) and thereby of Theorem 15 is complete.

Remark. If G is not bounded, a portion of the latter argument is still valid. By Theorem 10, every element of the null space of $(A_p - \zeta I)$ must lie in the null-space of $(A_r - \zeta I)$ for $r > p$, while if u is a solution of $(A_p - \zeta I)u = f$ with f in $R(G)$, then u lies in $D(A_r)$ and is a solution of $(A_r - \zeta I)u = o$. It follows that the point spectrum of $(A_p - \zeta I)$ increases with p, while the set of values of ζ for which $(A_p - \zeta I)$ has a non-dense range decreases.

Proof of Theorem 17. By Theorem 15(c), $(A_p - \zeta I)^{-1}$ will exist on a bounded domain G under our given hypotheses for ζ real and large negative. By Theorem 2, however, for every such ζ, $(A_p - \zeta I)^{-1}$ is not only a bounded mapping into $L^p(G)$ but also into $W^{2m,p}(G)$. By Lemma 5, every such mapping is a compact linear mapping of $L^p(G)$ for G bounded, and it follows by well-known arguments due essentially to F. RIESZ (cf. HILLE and PHILLIPS [34]) that the spectrum of A_p is discrete, the eigenspaces of finite dimension, and $(A_p - \zeta I)^{-1}$ will be defined on all of $L^p(G)$ whenever $(A_p - \zeta I)$ has a trivial nullspace (and by our preceding remarks, whenever $(A_2 - \zeta I)$ has a trivial null-space).

Proof of Theorem 16. By Theorem 15, A_2 has a point in its resolvent set for any bounded regular domain G. Applying Theorem 12, it follows that $(A_p)^* = A'_q$ for any p with $1 < p < \infty$, $q = p(p-1)^{-1}$, and more generally $(A_{p,0})^* = A'_{q,1}$, $(A_{p,1})^* = A'_{q,0}$. Thus (a) follows.

Let $A^{(s)} = 1/2\,(A + A')$. $A^{(s)}$ is a formally self-adjoint differential operator whose characteristic form is $s(x, \xi) = \sum_{|\alpha|=2m} r_\alpha(x)\xi^\alpha$, where $r_\alpha(x)$ is the real part of $a_\alpha(x)$. In particular $s(x, \xi) = \mathrm{Re}(a(x, \xi)) \geqq c_0|\xi|^{2m}$ for real n-vectors ξ by the uniform strong ellipticity of A, and $A^{(s)}$ is uniformly elliptic and indeed uniformly strongly elliptic on G. Since the coefficients of the characteristic form of $A^{(s)}$ are real, it follows that $A^{(s)}$ if regularly elliptic, since for every complex root λ of the equation $s(x, t + \lambda N) = 0$, $\bar{\lambda}$, the complex conjugate of λ, is also a root and therefore there must be exactly m roots (with multiplicities) in the upper half λ-plane. By Theorem 16(a), $(A_2^{(s)})^* = A_2^{(s)}$, i.e. $A_2^{(s)}$ is self-adjoint in the Hilbert space sense. On the other hand, by Theorem 15, the spectrum of $A_2^{(s)}$ lies on the half-line $\xi \geqq -K$. It follows easily from the spectral theory for self-adjoint operators that $A_2^{(s)}$ is bounded from below, i.e. $(A_2^{(s)}u, u) \geqq -K(u, u)$ for all u in $D(A_2)$. But $(A_2^{(s)}u, u) = \mathrm{Re}(Au, u)$, so that $\mathrm{Re}(Au, u) \geqq -K(u, u)$ for all u in $D(A_2)$.

If A is uniformly strongly elliptic and regularly elliptic however, so is $A - \varepsilon(-1)^m \Delta^m$, where ε is any suitably small constant such that $\varepsilon > 0$, and Δ is the ordinary Laplace operator. For this operator $A_{(\varepsilon)}$, we have by the preceeding argument $\mathrm{Re}(A_{(\varepsilon)}u, u) \geqq -K(u, u)$ for all u in $D(A_2)$. But $\mathrm{Re}(A_{(\varepsilon)}u, u) \leqq \mathrm{Re}(Au, u) - \varepsilon k\|u\|_m^2$, by the usual Fourier transform argument, so that

$$\mathrm{Re}(Au, u) \geqq c_1\|u\|_m^2 - K\|u\|^2,$$

and Theorem 16(c) is established.

Theorem 16(b) is an easy consequence of Theorem 16(c). Indeed, for $\xi \geqq K$, u in $D(A_2)$, $\zeta = \xi + i\eta$,

$$\|(A_2 + \zeta I)u\| \cdot \|u\| \geqq |\mathrm{Re}(A_2u, u) + \xi(u, u)| \geqq (\xi - K)\|u\|^2.$$

Dividing through by $\|u\|$,

$$\|(A_2 + \zeta I)u\| \geqq (\xi - K)\|u\|, \xi = \mathrm{Re}(\zeta).$$

The location of the spectrum follows from this last inequality as in the proof of Theorem 15, for we have as a consequence that $(A_2 - \zeta I)$ has a trivial null-space for $\mathrm{Re}(\zeta) \leqq -K$. Moreover, since for such ζ the range of $(A_2 - \zeta I)$ is the whole of $L^2(G)$, it follows that for $\xi \geqq K$

$$\|(A_2 + \xi I)u\| \geqq (\xi - K)\|u\|.$$

Thus the proof of Theorem 16 is complete.

Corollary to Theorem 16. *If A is a regularly elliptic, uniformly strongly elliptic operator on a bounded open subset G of E^n which is regular of class C^{4m} and if A satisfies the regularity condition* (IV) *on G, then there exists a constant K such that the half plane $\{\zeta : \mathrm{Re}\,\zeta \leqq -K\}$ lies in the resolvent set of A_p for all p with $1 < p < \infty$.*

Proof of the Corollary to Theorem 16. Our conclusion follows from the result on the spectrum of A_2 in Theorem 16 combined with our previous observation that for G bounded, all the operators A_p have the same spectrum.

Before proceeding to the sharpening of our results for the case when the operator A is essentially real, i.e. when the top-order coefficients of A are real-valued functions, let us note the following sharpened form of Theorem 17.

Theorem 18. *Let G be a bounded open subset of E^n which is regular of class C^{2m}, A a regularly elliptic, uniformly strongly elliptic differential operator on G which satisfies the regularity conditions* (I) *and* (II) *on its coefficients. Let $1 < p < \infty$. Then the operator $(A_p - \xi I)^{-1}$, which exists for ξ large and negative by Theorem* 15, *has the following property: Let $\xi \geqq K$, $s > n(2mp)^{-1}$. Then there exist functions $k_s(x, y, \xi)$ measurable on $G \times G$ for fixed ξ and constants M_s such that*

$$\{(A_p + \xi I)^{-s} f\}(x) = \int_G k_s(x, y, \xi) f(y)\, dy,\ f \in L^p(G),$$

$$\int_G |k_s(x, y, \xi)|^q\, dy \leqq M_s,\ x \text{ in } G,\ (q = p(p-1)^{-1}).$$

In particular if $p = 2$, $(A_2 + \xi I)^{-s}$ is an operator of Hilbert-Schmidt type if $s > n(4m)^{-1}$, and $(A_2 + \xi I)^{-2s}$ is an operator of finite trace class.

Proof of Theorem 18. By a preceding argument, $(A_p + \xi I)^{-1}$ coincides with $(A_r + \xi I)^{-1}$ on any function in $L^r(G)$ if $r \geqq p$. We know that $(A_p + \xi I)^{-1}$, which is a bounded mapping of $L^p(G)$ into itself for $\xi \geqq K$, must be a bounded linear mapping of $L^p(G)$ into $W^{2m,p}(G)$ by Theorem 2. By Lemma 5, $W^{2m,p}(G)$ is contained in $L^r(G)$ for $p \leqq r \leqq np(n-2mp)^{-1}$, $(n-2mp) > 0$, and the imbedding mapping is a bounded linear map. Hence $(A_p + \xi I)^{-1}$ is a bounded linear mapping of $L^p(G)$ into $L^r(G)$ for such r and for all r, if $n - 2mp = 0$. Moreover, if $(n-2mp) < 0$, then $W^{2m,p}(G)$ is continuously imbeddable in $C^0(\bar{G})$. It follows in that case that $(A_p + \xi I)^{-1}$ is a bounded linear mapping of $L^p(G)$ into $L^\infty(G)$.

Since the exponent r is defined by $r^{-1} = p^{-1} - 2mn^{-1}$, whenever the latter is positive, any number $\geqq p$ when the right side is zero, and ∞ when the right side is negative, it follows from the coincidence of the inverses of $(A_p + \xi I)^{-1}$ for different values of p on their common domain, that $(A_p + \xi I)^{-s}$ maps $L^p(G)$ into $L^r(G)$, where $r^{-1} \geqq p^{-1} - 2msn^{-1}$. If s is greater than $n(2mp)^{-1}$, we have, $p^{-1} - 2msn^{-1} < 0$ and $(A_p + \xi I)^{-s}$ is a bounded linear mapping of $L^p(G)$ into $L^\infty(G)$. Applying the Dunford-Pettis Theorem ([23]), the existence of functions $k_s(x, y, \xi)$ satisfying the conditions of Theorem 18 follows. For $p = 2$, any integral operator T with a kernel $k(x, y)$ such that

$$\int_{G \times G} |k(x, y)|^2\, dx\, dy < \infty$$

is an operator of Hilbert-Schmidt type, and T^2 is an operator of finite trace class. Applying this to the operator $(A_2 + \xi I)^{-s}$, we obtain the final conclusions of Theorem 18.

Theorem 19. *Suppose that A is a regularly elliptic, uniformly strongly elliptic differential operator defined on an open subset G of E^n, with G uniformly regular of class C^{2m}. Suppose that A is essentially real on G and that A satisfies*

the regularity conditions (I) *and* (II) *on* G. *Then given* ε *with* $0 < \varepsilon < 1$, *there exists a constant* $K(\varepsilon) > 0$ *such that for all complex* ζ *with* $\mathrm{Re}(\zeta) \geqq 0$, *and* u *in* $D(A_2)$,

$$\|(A_2 + \zeta I)u\| \geqq \{(1-\varepsilon)|\zeta| - K(\varepsilon)\}\|u\| . \tag{4.25}$$

In particular, if G *is bounded, then the spectrum of* A_p *for all* p *with* $1 < p < \infty$ *lies in a half-plane of the form* $\{\zeta : \mathrm{Re}(\zeta) \geqq K_1\}$.

Theorem 20. *Suppose that* A *is a regularly elliptic, uniformly elliptic differential operator defined on an open subset* G *of* E^n, *with* G *uniformly regular of class* C^{2m} *and locally regular of class* C^{4m}. *Suppose that* A *is essentially real on* G, *and that* A *satisfies the regularity condition* (IV) *on* G. *Then:*

(a) *There exists a constant* K *such that for all* u *in* $D(A_2)$ *and all complex* ζ *with* $\mathrm{Re}(\zeta) \geqq 0$,

$$\|(A_2 + \zeta I)u\| \geqq \{|\zeta| - K|\zeta|^{\lambda_1} - K\}\|u\| \tag{4.26}$$

with $\lambda_1 = (2m-1)(2m+1)^{-1}$.

(b) *There exists a constant* K_1 *such that the spectrum of* A_2 *is contained in the set*

$$\{\zeta : \mathrm{Re}(\zeta) \geqq -K_1,\ |\mathrm{Im}(\zeta)| \leqq K_1(K_1 + \mathrm{Re}(\zeta))^{\lambda_2}\} ,$$

with $\lambda_2 = (2m-1)(2m)^{-1}$.

(c) *For* $\mathrm{Re}(\zeta) \geqq 0$, $|\zeta| > K_2$, *we have an inequality of the form*

$$\|(A_2 + \zeta I)^{-1}\| \leqq (|\zeta| - K_2|\zeta|^{\lambda_1})^{-1} .$$

Proof of Theorem 19. Following the pattern of the proof of Theorem 15, for $\mathrm{Re}(\zeta) \geqq 0$, we consider for u in $D(A_2)$,

$$\|(A_2 + \zeta I)u\|^2 = \|A_2 u\|^2 + |\zeta|^2\|u\|^2 + 2\,\mathrm{Re}(\bar{\zeta}(A_2 u, u)) .$$

Decomposing $(A_2 u, u)$ as before, we have

$$(A_2 u, u) = R(u) + \sum_k (B_k(\eta_k u), \eta_k u)$$

where B_k is a homogeneous operator with constant coefficients obtained by evaluating the top-order coefficients of A at a point x_k of the neighborhood N_k of a certain covering of G chosen in the proof of Theorem 15, and where $R(u)$ satisfies the inequality

$$|R(u)| \leqq K\|u\|\Big\{\sum_{0 \leqq r \leqq 2m-1} \delta^{r-2m}\|u\|_r + w(\delta)\,\|u\|_{2m}\Big\}$$

for a δ with $0 < \delta < 1$, ($w(\delta)$ being the sum of the δ-oscillations of the top-order coefficients of A).

If A is essentially real, so are the differential operators B_k, so that for each k, $(B_k(\eta_k u), \eta_k u) \geqq 0$. It follows that

$$\begin{aligned} \mathrm{Re}(\bar{\zeta}(A_2 u, u)) &\geqq -|\zeta|\,|R(u)| \geqq -K|\zeta| \times \\ &\times \|u\|\Big\{\sum_{0 \leqq r \leqq 2m-1} \delta^{r-2m}\|u\|_r + w(\delta)\,\|u\|_{2m}\Big\} . \end{aligned} \tag{4.27}$$

The remainder of the proof of (4.25) is identical with the proof of (4.1) except that ξ is now replaced by $|\zeta|$. The identity of the spectra of A_p for bounded G, and the conclusion that they all lie in a half-plane follow easily from (4.25) together with the fact that $(A_2 - \zeta I)^{-1}$ will exist on a bounded domain if $(A_2 - \zeta I)$ has a trivial null-space.

Proof of Theorem 20. Let $A^{(s)}$ be the symmetrized form of A, as in the proof of Theorem 16. If A is essentially real, then $A = A^{(s)} + S$, with S a differential operator of order $< 2m$. We have therefore for $\mathrm{Re}(\zeta) \geqq 0$,

$$\|(A_2 + \zeta I)u\| \geqq \|(A_2^{(s)} + \zeta I)u\| - \|Su\|\,, \tag{4.28}$$

and

$$\begin{aligned}\|(A_2^{(s)} + \zeta I)u\|^2 &= \|A_2^{(s)}u\|^2 + |\zeta|^2\|u\|^2 + 2\,\mathrm{Re}(\zeta(A_2^{(s)}u, u)) \geqq \\ &\geqq \|A_2^{(s)}u\|^2 + |\zeta|^2\|u\|^2 \geqq c_0\|u\|_{2m}^2 + (|\zeta|^2 - K)\|u\|^2\,,\end{aligned} \tag{4.29}$$

by Theorem 2, and the fact that $(A_2^{(s)}u, u) \geqq 0$, while $\mathrm{Re}(\zeta) \geqq 0$.

Combining (4.28) and (4.29), and noting that $\|Su\| \leqq K_1\|u\|_{2m-1}$, we have

$$\frac{\|(A_2 + \zeta I)u\|}{\|u\|} \geqq (c_0 a^2 + (|\zeta|^2 - K))^{1/2} - K_1 a^{(2m-1)/2m}, \tag{4.30}$$

with $a = \|u\|_{2m} \cdot \{\|u\|\}^{-1}$. We shall establish (4.26) by finding the minimum of the right-hand side of (4.30) over all positive values of a. Let $f(a) = (c_0 a^2 + c_1)^{1/2} - K_1 a^r$, with $c_1 \geqq 0$, $c_0 > 0$, and $0 < r < 1$. For $a = 0$, $f(a) = c_1^{1/2}$, while as $a \to \infty$, $f(a) \to \infty$. Unless the minimum of $f(a)$ is $c_1^{1/2}$, it is assumed at a stationary point, i.e. $f'(a) = 0$. At such a point, $c_0 a^{2-r} = K_1 r\{c_0 a^2 + c_1\}^{1/2}$. As a result, we have for such a,

$$c_0 a^{2-r} \leqq K_1 r\{c_0^{1/2} a + c_1^{1/2}\}\,,$$

from which it follows that

$$a^{2-r}\{c_0 - (K_1 r\, c_0^{1/2} a^{1-r})\} \leqq K_1 r\, c_1^{1/2}.$$

Thus, either $a \leqq K(K_1, r, c_0)$, or $a \leqq K\, c_1^{(4-2r)^{-1}}$, $K > 0$. Setting $c_1 = (|\zeta|^2 - K)$, and assuming that $|\zeta|^2 > K$, we see that if a is bounded, (4.30) will imply that $\|(A_2 + \zeta I)u\| \geqq \{|\zeta| - K_3\}\|u\|$. Thus, we need only consider the case when $a \leqq K\, c_1^{(4-2r)}$. But in that case,

$$f(a) \geqq c_1^{1/2} - K\, c_1^{r/(4-2r)}.$$

Putting in the values, $c_1 = |\zeta|^2 - K$, and $r = 2m - 1/2m$, we obtain

$$\|(A_2 + \zeta I)u\| \geqq \{|\zeta| - K_4|\zeta|^{(2m-1)(2m+1)^{-1}}\}\|u\|\,, \tag{4.31}$$

which is precisely the same as (4.26).

The same argument may be applied to A_2' and yields an inequality parallel to (4.26). The inequality (4.26) itself implies that for $\mathrm{Re}(\zeta) \geqq 0$ and $|\zeta|$ large, $(A_2 + \zeta I)$ has a trivial null-space and a closed range. The orthogonal complement of this range, which by Theorem 16 is precisely the null space of $(A_2' + \bar{\zeta} I)$, is trivial by the corresponding inequality for A_2'. Thus the range of $(A_2 + \zeta I)$ is all of $L^2(G)$, and $(A_2 + \zeta I)^{-1}$ is defined as a bounded operator on all of $L^2(G)$. In particular, (4.26) implies the inequality of (c) and Theorem 20 (c)

is proved. Finally, Theorem 20(b) is a consequence of an elementary computation using Theorem 20(c) and the first resolvent equation, as we have applied it above in the proofs of Theorems 15 and 16.

Section 5: Operators with potentials

In all the preceding discussion, we have considered only operators A whose coefficients satisfy the boundedness condition (I). In the present section, we extend these results to operators of the form $A + V(x)I$, where A is one of the previously considered operators while V, the so-called potential function, is a complex valued function defined on G but not necessarily bounded. The prototype of such operators is, of course, the Schroedinger operator $-\triangle + V(x)I$. A separate and more detailed discussion of second-order operators is given in Section 7.

We consider two kinds of potential functions V, the first having possible singularities in the finite part of G but of a restricted type, the second being locally in L^r for a suitable constant r and bounded from below on all of G.

Definition 8. *$V(x)$ is a potential of type s, $(1 < s < \infty)$ if*

$$\sup_{x \in G} \int_{G \cap \{y : |x-y| \leqq r\}} |x-y|^{2ms-n-a} |V(y)|^s \, dy = o(1), \; r \to 0 \,,$$

for some positive constant a, while V is locally in L^p on a neighborhood of each point of the closure of G.

Definition 8a. *$V(x)$ is a regular potential if V lies in $L^r(N)$ for some neighborhood N of each point of $G \cup \Gamma$, $r > \min\{2, n/2m\}$, while $\mathrm{Re}\{V(x)\} \geqq -c$ for some constant c for all x in G.*

Definition 9. *If $V(x)$ is a potential function on G, A an elliptic operator of order $2m$ on G, let $D(A_p + V(x)I)^{\#} = \{u : u \in R(G) \cap D(A_p), Au + Vu \in L^p(G)\}$, $D(A_p + V(x)I) = \{u : u \in D(A_p), Au + Vu \in L^p(G)\}$,*

$$(A_p + V(x)I)^{\#} u = Au + V(x)u, \; u \in D(A_p + V(x)I)^{\#},$$

$$(A_p + V(x)I)\, u \;\; = Au + V(x)u, \; u \in D(A_p + V(x)I).$$

Theorem 21. *Let $V(x)$ be a potential of type p, $1 < p < \infty$, on an open subset of E^n which is uniformly regular of class C^{2m} and locally regular of class C^{4m}. Let A be a regularly elliptic, uniformly strongly elliptic differential operator on G which satisfies the regularity condition* (IV). *Then:*

(a) *The operator $(A_p + V(x)I)^{\#}$ in $L^p(G)$ is closable, and its closure is $(A_p + V(x)I)$. The latter operator is closed, and its domain is equal to $D(A_p)$.*

(b) *If in addition $V(x)$ is a potential of type q for $q = p(p-1)^{-1}$ and if $\overline{V}$ is the complex conjugate of V, then $(A_p + V(x)I)^* = (A'_q + \overline{V}(x)I)$.*

(x) *If $p = q = 2$, then $\mathrm{Re}(A_2 u + Vu, u) \geqq K(u, u)$ for some constant K and all u in $D(A_2 + V(x)I) = D(A_2)$. If $V(x)$ is real-valued and A is formally self-adjoint, then $(A_2 + V(x)I)$ is self-adjoint and semi-bounded, while $(A_2 + V(x)I)^{\#}$ is essentially self-adjoint.*

The proof of Theorem 21 uses the following proposition, which is only proved for $p = 2$ in the present paper for general m (Theorem 16) and for all p

for second-order operators in Section 7. The proof for general m and p will be given in the second paper of this series.

Proposition 1. *Let A be a regularly elliptic, uniformly strongly elliptic differential operator of order $2m$ on an open subset G of E^n, with G uniformly regular of class C^{2m} and locally regular on class C^{4m}. Suppose that A satisfies the regularity condition* (IV) *of Section* 3. *Then for ξ sufficiently large and positive, $(A_p + \xi I)^{-1}$ will exist on all of $L^p(G)$, and*

$$C_0 \|(A_p + \xi I)u\|_{0,p} \geqq \|u\|_{2m,p} + \xi \|u\|_{0,p} ,$$

for large positive ξ.

Another preliminary result which is used in the proof of Theorem 21 is contained in the following Lemma:

Lemma 16. *Let V be a potential of type p, $1 < p < \infty$. Then for u in $W^{2m,p}(G)$, Vu lies in $L^p(G)$, and given $\varepsilon > 0$, there exists a constant $K(\varepsilon)$ such that for all u in $W^{2m,p}(G)$,*

$$\|V(x)u\|_{0,p} \leqq \varepsilon \|u\|_{2m,p} + K(\varepsilon)\|u\|_{0,p} . \tag{5.1}$$

Proof of Lemma 16. We may restrict ourselves without loss of generality to functions u in $R(G)$, provided that we establish (5.1) with a constant $K(\varepsilon)$ uniformly for all such u. We shall derive (5.1) from another inequality,

$$|u(x)|^p \leq K(a) r^a \int_{G \cap |x-y| \leq r} |x-y|^{2mp-n-a} \sum_{|\alpha|=2m} |D^\alpha u(y)|^p dy + K(a) r^{a-2mp} \int_{G \cap |x-y| \leq r} |x-y|^{2mp-n-a} |u(y)|^p dy , \tag{5.2}$$

which is valid for any point x in G, $0 < a < 1$, and any $r > 0$ which is smaller than the radius of the standard Sobolev cone in G. The inequality (5.2) implies, indeed, if we multiply it by $|V(x)|^p$ and integrate in x over G, that

$$\|V(x)u\|^p_{L^p(G)} \leqq K(a) r^a \left(\int_{|x-y| \leq r} |V(x)|^p \cdot |x-y|^{2mp-n-a} dx \right) \cdot \sum_{|\alpha|=2m} \|D^\alpha u\|^p_{0,p} + K(a) r^{a-2mp} \left(\int_{|x-y| \leq r} |x-y|^{2mp-n-a} |V(x)|^p dx \right) \|u\|^p_{0,p} . \tag{5.3}$$

Since each of the integrals in (5.3) is finite and decreases with decreasing r, we may by choosing r sufficiently small make sure that (5.1) holds for any given $\varepsilon > 0$. To prove Lemma 16, it therefore suffices to establish (5.2).

For any point x of G, it follows from the uniform regularity of G that G is a Sobolev domain and hence there exists $h > 0$, independent of x, and a spherical cone of radius h with vertex at x and a base independent of x which lies completely in G. Let C_x be such a cone. Suppose that we establish (5.2) for $r = h$, and the domain $G \cap |x-y| \leq r$ replaced by C_x, where x is the vertex of C_x, i.e. suppose that we prove the inequality

$$|u(x)|^p \leqq K \int_{C_x} |x-y|^{2mp-n-a} \sum_{|\alpha|=2m} |D^\alpha u(y)|^p dy + K \int_{C_x} |u(y)|^p dy . \tag{5.4}$$

Then changing the variable of integration by the substitution $(y-x) = rh^{-1}(\xi - x)$, under which the cone C_x passes into a subset of $G \cap |x-y| \leq r$, we obtain (5.2). It therefore suffices to prove (5.4).

To prove (5.4), let $0 < h_1 < h$, and C' be the cone with vertex at x, radius h_1, and with the same axis as, and similar to, C_x. Let $\lambda(r)$ be a function from $C_c^\infty(E^1)$ with $\lambda(r) = 1$ for $r \leqq h_1$; $\lambda(r) = 0$ for $r \geqq h$; $0 \leqq \lambda(r) \leqq 1$. Every point y of C_x can be written in the form $y = x + r\xi$, with $0 \leqq r \leqq h$, ξ lying in a fixed set S on the unit $(n-1)$-dimensional sphere, while for points y in C', $0 \leqq r \leqq h_1$. We consider the integral

$$\int_{C'} |x-y|^k |v(y)|^p dy \leqq \int_S \int_0^h \lambda(r) r^k |v(x+r\xi)|^p r^{n-1} dr\, dS_\xi ,$$

for $k > -n$. Integrating by parts, we obtain,

$$(5.5)\quad \int_0^h \lambda(r) r^k |v(x+r\xi)|^p r^{n-1} dr \leqq (n+k)^{-1} \int_0^h r^{n+k} \{ |\lambda_r| \, |v(x+r)|^p + p\lambda(r) \, |v|^{p-1} \, |v_r| \} dr .$$

Integrating (5.5) with respect to dS_ξ over the subset S of the unit $(n-1)$ sphere, and using the usual inequality $ab < \varepsilon p^{-1} a^p + \varepsilon^{-(p-1)^{-1}} (p-1) b^{p(p-1)^{-1}}$ to estimate the second integral on the right in (5.5) in terms of the left-side, we obtain

$$(5.6)\quad \int_{C'} |x-y|^k \, |v(y)|^p \, dy \leqq K \int_{C \cap \{h_1 \leqq r \leqq h\}} |v(y)|^p \, dy + \int_C |x-y|^{k+p} \sum_j |D_j v|^p \, dy$$

for any v in $R(G)$, (with the constant K depending on h on h_1). Iterating (5.6) on a sequence of increasing cones, we obtain the inequality,

$$(5.7)\quad \int_{C'} |x-y|^k \, |v(y)|^p \, dy \leqq K \int_C |x-y|^{k+sp} \sum_{|\alpha|=s} |D^\alpha v|^p \, dy + \int_{C \cap \{h_1 \leqq r \leqq h\}} \sum_{|\beta| < s} |D^\beta v|^p \, dy .$$

But since the domain $C \cap \{h_1 \leqq r \leqq h\}$ is itself a Sobolev domain, we have the inequality

$$(5.8)\quad \int_{C \cap \{h_1 \leqq r \leqq h\}} \sum_{|\beta| < s} |D^\beta v|^p \, dy \leqq K \int_{C \cap \{h_1 \leqq r \leqq h\}} \Big\{ |v|^p + \sum_{|\alpha|=s} |D^\alpha v|^p \Big\} dy \leqq \leqq K_1 \int_C |x-y|^k \Big\{ |v(y)|^p + \sum_{|\alpha|=s} |D^\alpha v(y)|^p \Big\} dy$$

for any $k > -n$. If we apply (5.7) and (5.8) to a derivative $D^\beta u$ of our original function u from $R(G)$, again estimating the L^p-norm of $D^\beta u$ on $C \cap \{h_1 \leqq r \leqq j\}$ in terms of the L^p-norms of higher and lower order derivatives on the same set, we obtain

$$(5.9)\quad \int_{C'} |x-y|^{sp-n-a} \, |D^\beta u(y)|^\beta \, dy \leqq \leqq K_2 \int |x-y|^{2mp-n-a} \Big\{ |u(y)|^p + \sum_{|\alpha|=2m} |D^\alpha u(y)|^p \Big\} dy .$$

Let $\lambda_1(r)$ be another localizing function, like the $\lambda(r)$ chosen above, with $\lambda_1(r) = 0$ for $r \geqq h_1$, $\lambda_1(r) = 1$ for r close to zero. We have,

$$(5.10)\quad u(x) = -\int_0^{h_1} \frac{d}{dr} (\lambda_1(r) \, u(x+r\xi)) \, dr ,$$

and integrating with respect to dS_ξ over S, we have

$$|u(x)| \leqq K \int_{C'} r^{1-n} \left\{ |u(y)| + \sum_j |D_j u(y)| \right\} dy . \tag{5.11}$$

Let $0 < a < 1$. Applying Hölder's inequality to (5.11), we get

$$|u(x)|^p \leqq K_3(a) \int_{C'} |x-y|^{-n-a+p} \left\{ |u(y)|^p + \sum_j |D_j u(y)|^p \right\} dy . \tag{5.12}$$

Applying (5.9), we obtain

$$|u(x)|^p \leqq K_4(a) \int_{C'} |x-y|^{2mp-n-a} \left\{ \sum_{|\alpha| = 2m} |D^\alpha u(y)|^p + |u(y)|^p \right\} dy , \tag{5.13}$$

which is identical with (5.4), and thereby Lemma 16 is proved.

Proof of Theorem 21. Since $V(x)$ lies locally in L^p on a neighborhood of each point of the closure of G, it follows that the domain of $(A_p + V(x)I)^\#$ is all of $D(A_p) \cap R(G)$.

Since all the properties to be established in Theorem 21, as well as the hypotheses, are invariant under the addition of a constant multiple of the identity operator I in $L^p(G)$ to A_p, we may replace our given operator A_p by $A_p + \xi I$, with ξ large and positive. By Lemma 16, for u in $D(A_p) \cap R(G)$, we have

$$\| Vu \|_{0,p} \leqq \varepsilon \| u \|_{2m,p} + K(\varepsilon) \| u \|_{0,p} .$$

Let $v = (A_p + \xi I)u$. Then by Proposition 1,

$$\| u \|_{2m,p} + \xi \| u \|_{0,p} \leqq c_0 \| v \|_{0,p} . \tag{5.14}$$

Combining (5.1) with (5.14), we obtain

$$\| V(x) (A_p + \xi I)^{-1} v \| \leqq \{ \varepsilon c_0 + K(\varepsilon) c_0 \xi^{-1} \} \| v \|_{0,p} . \tag{5.15}$$

Choosing ε so small that $\varepsilon c_0 < 4^{-1}$, and then ξ so large that $K(\varepsilon) c_0 \xi^{-1} < 4^{-1}$, we obtain for such large ξ from (5.15) that

$$\| V(x) (A_p + \xi I)^{-1} v \|_{0,p} \leqq 2^{-1} \| v \|_{0,p} \tag{5.16}$$

for all v in the image of $D(A_p) \cap R(G)$ under $(A_p + \xi I)$.

Let $(A_p + I)^\#$ be the restriction of $(A_p + \xi I)$ to $D(A_p) \cap R(G)$. It follows from our previous results that the closure of $(A_p + \xi I)^\#$ is $A_p + \xi I$. For u in $D(A_p) \cap R(G)$, we have, however,

$$(A_p + \xi I + V(x)I)u = (I + V(x) (A_p + \xi I)^{-1}) (A_p + \xi I)^\# u . \tag{5.17}$$

Since the operator $R = V(x) (A_p + \xi I)^{-1}$ has norm less than 2^{-1}, $(I + R)$ is an invertible operator with a bounded inverse. Suppose that for a sequence u_n in $D(A_p) \cap R(G)$, we have $u_n \to u_0$ in $L^p(G)$, $(A_p + V(x)I)^\# u_n \to w$ in $L^p(G)$. Then $(A_p + \xi I + V(x)I)^\# u_n \to \xi u_0 + w_0$, while $(A_p + \xi I)^\# u_n = (I + R)^{-1} (\xi u_n + w_n)$ converges to $(I + R)^{-1}(\xi u_0 + w_0)$. Since $(A_p + \xi I)$ is a closed operator, it follows that u_0 lies in $D(A_p)$. Since $(A_p + \xi I)u_n$ converges, it follows that $V(x)u_n$ converges to v_0 in $L^p(G)$. Since a subsequence of $V(x)u_n$ converges a.e. to v_0 in G, while (V being finite a.e.) a subsequence of u_n converges to u_0 a.e. and hence $V(x)u_n$ converges a.e. to $V(x)u_0$ in G, it follows that

$V(x)u_0 = v_0$, and u_0 lies in $D(A_p + V(x)I)$. Moreover, $\lim_n (A_p + \xi I + V(x)I)^{\#} u_n = (A_p + \xi I + V(x)I)u_0$. Thus $(A_p + V(x)I)^{\#}$ is closable and its closure is contained in $(A_p + V(x)I)$.

Suppose on the other hand that u_0 is any element of $D(A_p)$. Since the closure of $(A_p + \xi I)^{\#}$ is $(A_p + \xi I)$, there is a sequence $\{u_n\}$ from $D(A_p) \cap R(G)$ such that $u_n \to u_0$, $(A_p + \xi I)u_n \to (A_p + \xi I)u_0$, both in $L^p(G)$. By (5.17), $(A_p + \xi I + V(x)I)u_n$ converges in $L^p(G)$, so that u_0 lies in the closure of the operator $(A_p + V(x)I)^{\#}$. We have already verified that the closure of $(A_p + V(x)I)^{\#}$ is contained in $(A_p + V(x)I)$, so that it follows that $D(A_p) = D(A_p + V(x)I)$, and that the closure of $(A_p + V(x)I)^{\#}$ is precisely $(A_p + V(x)I)$. Thus part (a) of Theorem 21 is proved.

To prove (b), suppose that v lies in the domain of $(A_p + V(x)I)^*$. It follows from part (a) that if V is a potential of type q, then $(A' + \overline{V}(x)I)$ is a closed operator in $L^q(G)$. Moreover, for u in $D(A_p)$, w in $D(A'_q)$, $(A_p u + V(x)u, w) = (A_p u, w) + (Vu, w) = (u, A'_q w) + (u, \overline{V}w) = (u, A'_q w + \overline{V}(x)w)$, so that $(A'_q + \overline{V}(x)I) \subseteq (A_p + V(x)I)^*$. We need prove only the reverse inclusion. Let $f = (A_p + V(x)I)^* v$, $f \in L^q(G)$. Then,

$$(A_p u + \xi u + V(x)u, v) = (u, f + \xi v) \tag{5.18}$$

for all u in $D(A_p)$. By (5.17), $(A_p + \xi I + V(x)I) = (I + R)(A_p + \xi I)$, with R a bounded operator of norm less than 2^{-1}, $(I + R)$ invertible.

Thus (5.18) becomes,

$$((I + R)(A_p + \xi I)u, v) = ((A_p + \xi I)u, (I + R^*)v) = (u, (f + \xi v)) . \tag{5.19}$$

Thus $(I + R^*)v$ lies in the domain of $(A_p + \xi I)^*$, which by Theorem 16 equals $(A'_q + \xi I)$, and $(A'_q + \xi I)(I + R^*)v = f + \xi v$. Since $\|R^*\| = \|R\| < 2^{-1}$, $(I + R^*)$ maps $L^q(G)$ bicontinuously onto itself. We have verified that if v lies in the domain of $(A_p + V(x)I)^*$, then $(I + R^*)v$ lies in $D(A'_q + \xi I)$. The converse follows by the same argument, so that $D(A_p + V(x)^*) = (I + R^*)^{-1}\{D(A'_q + \xi I)\}$; $(A'_q + \xi I)(I + R^*) = (A_p + V(x)I)^* + \xi I$.

Since both $(A'_q + \xi I)$ and $(I + R^*)$ have continuous inverses defined on all of $L^q(G)$, it follows from the last equality that the linear transformation $(A_p + V(x)I)^* + \xi I$ in $L^q(G)$ is one-to-one and has a bounded inverse defined on all of $L^q(G)$ for large positive ξ. By our previous remarks, $(A'_q + \xi I + \overline{V}(x)I) \subseteq \subseteq (A_p + V(x)I)^* + \xi I$. Since $V(x)$ is a potential of type q, however, we may apply the argument which let to (5.17) to the operator $(A'_q + \overline{V}(x)I)$ in $L^q(G)$, instead of $A_p + V(x)I$ in $L^p(G)$, and we obtain

$$(A'_q + \overline{V}(x)I + \xi I) = (I + R_1)(A'_q + \xi I) , \tag{5.20}$$

with R_1 a transformation of norm less than 2^{-1} for large positive. It follows from (5.20), since both of the transformations on the right side have bounded inverses defined on all of $L^q(G)$, that $(A'_q + \overline{V}(x) + \xi I)$ has a bounded inverse defined on all of $L^q(G)$ for large positive ξ. But, two transformations T_1 and T_2, both one-to-one and onto, for which $T_1 \subseteq T_2$ must be equal. It follows then that $(A_p + V(x)I)^* + \xi I = (A'_q + \overline{V}(x)I) + \xi I$, and consequently $(A_p + V(x)I)^* = A'_q + \overline{V}(x)I$. Thus, part (b) of Theorem 21 is established.

Part (c) of Theorem 21 is a simple consequence of part (b). Let H be the symmetric part of A, i.e. $H = 2^{-1}(A + A')$. H satisfies the hypotheses of Theorem 21 on A, and hence if V_1 is the real part of the function V, $H_2 + V_1(x)I$ is self-adjoint in $L^2(G)$, and its domain is $D(H_2) = D(A_2) = D(A_2 + V(x)I)$. By our preceding argument, moreover, $H_2 + V_1(x)I + \xi I$ has a bounded inverse on all of $L^2(G)$ for large positive ξ. It follows immediately that the spectrum of the self-adjoint operator $H_2 + V_1(x)I$ is bounded from below and that, $(H_2 u + V_1(x)u, u) \geqq -K(u, u)$ for all u in $D(A_2 + V(x)I) = D(H_2 + V_1(x)I)$, and some fixed constant K. But for such u, $\mathrm{Re}\big((A_2 + V(x)I)u, u\big) = \big((H_2 + V_1(x)I)u, u\big) \geqq -K(u, u)$, and the first assertion of (c) is established. The remaining assertions of (c) are special cases of those of (a) and (b).

The argument of the proof of Theorem 21 may be applied in a more abstract context to obtain the following result, which we give explicitly to simplify the exposition of the next concrete application of the same method of proof.

Theorem 22. *Let X be a Banach space, X^* its adjoint space, A a densely defined closed operator in X, A^* its adjoint which we assume densely defined in X^*. Let V and V' be linear operators with domains containing the sets D and D' in X and X^*, respectively, and ranges in the same spaces, such that the following conditions are satisfied:*

(a) *D is contained in $D(A)$, and the closure of A restricted to D is the operator A itself. D' is contained in $D(A^*)$, and the closure of the restriction of A^* to D' is A^* itself.*

(b) *There exists a complex number ζ such that $(A + \zeta I^{-1})$ exists on X, while for u in D,*

$$\|(A + \zeta I)u\| \geqq (1 + \delta)\|Vu\|, \quad (\delta > 0 \text{ independent of } u).$$

Similarly, we assume that for v in D',

$$\|(A^* + \xi I)v\| \geqq (1 + \delta)\|V'v\|.$$

(c) *The operators $V(A + \zeta I)^{-1}$ and $V'(A^* + \xi I)^{-1}$ are closed. (Note that $(A^* + \xi I)^{-1}$ exists as a simple consequence of* (b).*)*

(d) *For u in D, v in D',*

$$(Vu, v) = (u, V'v).$$

Let the operators $A + V$, $(A + V)^{\#}$, $A^* + V'$, and $(A^* + V')^{\#}$ be defined analogously to Theorem 21 *as follows:*

$$D((A + V)^{\#}) = D,\ D((A^* + V')^{\#}) = D',$$

$$D(A + V) = D(A) \cap D(V),\ D(A^* + V') = D(A^*) \cap D(V').$$

$$(A + V)u = Au + Vu \quad \text{for} \quad u \in D(A + V);$$

$$(A + V)^{\#} = (A + V)|D((A + V)^{\#}).$$

$$(A^* + V')v = A^*v + V'v,\ v \in D(A^* + V');$$

$$(A^* + V')^{\#} = (A^* + V')|D((A^* + V')^{\#}).$$

Then:

(i) *The closure of* $(A+V)^{\#}$ *is* $A+V$. *The closure of* $(A^*+V')^{\#}$ *is* A^*+V', $(A+V+\zeta I)^{-1}$ *exists on all of* X, *and* $(A^*+V'+\bar{\zeta}I)^{-1}$ *exists on all of* X^*.

(ii) $(A+V)^* = A^*+V'$.

Proof of Theorem 22. Proceeding as in the proof of Theorem 21, we have for u in D,

$$(A+V)^{\#}u+\zeta u = (I+V(A+\zeta I)^{-1})(A+\zeta I)^{\#}. \tag{5.21}$$

If $R = V(A+\zeta I)^{-1}$, assumption (b) implies that $\|R\| \leqq (1+\delta)^{-1} < 1$. Hence $(I+R)$ is a bounded invertible transformation in X. By (5.21), $(A+V)^{\#}+\zeta I$ is one-to-one and has a bounded inverse. Hence $(A+V)^{\#}+\zeta I$ is closable as a linear transformation in X. Let $\{u_n\}$ be a sequence from D with $u_n \to u_0$. It follows from (5.21), since both $(I+R)$ and $(I+R)^{-1}$ are bounded, that the conditions, $\{(A+V)^{\#}+\zeta I\}u_n$ converges, and $(A+\zeta I)u_n$ converges, are equivalent. Since $(A+\zeta I)$ is the closure of its restriction to D, it follows that the domain of the closure of $(A+V)$ is precisely $D(A)$. On the other hand, suppose $u_n \to u$, u in $D(A)$, $(A+\zeta I)u_n \to (A+\zeta I)u$. Then by (c), the sequence $V(A+\zeta I)^{-1}(A+\zeta I)u_n$ converges in X, and by (c), it converges to $V(A+\zeta I)\{(A+\zeta I)u\} = Vu$. Hence, $D(A) = D(A+V)$, and it follows immediately that the closure of $(A+V)^{\#}$ is $(A+V)$. The same argument establishes (i) for (A^*+V').

To prove (ii), we remark first that $(A^*+V') \subseteq (A+V)^*$ by (d). To prove the reverse inclusion, suppose v lies in $D((A+V)^*)$. Then for all u in $D(A) \cap D(V)$, (if $f = (A+V)^*v$),

$$((A+V+\zeta I)u, v) = (u, f+\bar{\zeta}v). \tag{5.22}$$

Substituting (5.21) in (5.22), we obtain,

$$((A+\zeta I)u, (I+R^*)v) = (u, f+\bar{\zeta}v),\ u \in D(A). \tag{5.23}$$

Conversely, (5.23) implies (5.22). However, (5.23) is equivalent to the assertion that $(I+R^*)v \in D(A^*+\bar{\zeta}I)$, or, since $(I+R^*)^{-1}$ is everywhere defined and bounded, to $v \in (I+R^*)^{-1}D(A^*+\bar{\zeta}I)$. In addition

$$(A+V)^*v = f = (A^*+\bar{\zeta}I)(I+R^*)v - \bar{\zeta}v, \tag{5.24}$$

or

$$(A+V)^*+\bar{\zeta}I = (I+R^*)(A^*+\bar{\zeta}I). \tag{5.25}$$

Since each of the transformations on the right has a continuous inverse defined on all of X^*, so has $(A+V)^*+\bar{\zeta}I$.

On the other hand, $(A^*+V'+\bar{\zeta}I) = (I+V'(A^*+\bar{\zeta}I)^{-1}(A^*+\bar{\zeta}I) = (I+R_1)(A^*+\bar{\zeta}I)$, with R_1 of norm less than one. In particular, $(A^*+V'+\bar{\zeta}I)$, being the composition of two invertible transformations onto X^*, itself has an inverse defined on all of X^*. However, since $(A^*+V'+\bar{\zeta}I) \subseteq ((A+V)^*+\bar{\zeta}I)$ and both have bounded inverses defined on all of X^*, $(A^*+V'+\bar{\zeta}I) = (A+V)^*+\bar{\zeta}I$, and hence $(A+V)^* = A^*+V'$, and Theorem 22 is proved.

To generalize Theorem 21 using Theorem 22, we shall consider a pertubation of our original differential operator A by a differential operator of order less than $2m$ with possibly singular coefficients.

Theorem 23. *Let A be a regularly elliptic, uniformly strongly elliptic differential operator of order $2m$ on an open subset G of E^n, with G uniformly regular of class C^{2m} and locally regular of class C^{4m}. We suppose that A satisfies the regularity condition* (IV), *A' the formal adjoint of A. Let B and B' be two differential operators of order $< 2m$, $B = \sum_{|\beta|<2m} b_\beta(x)D^\beta$, $B' = \sum_{|\beta'|<2m} b'_\beta(x)D^\beta$, where B' is the formal adjoint of B, and the coefficients $b_\beta(x)$ and $b'_\beta(x)$ lie locally in L^p on a neighborhood in G of each point of the closure of G and satisfy the conditions:*

$$\sup_{x\in G} \int_{G\cap(y:|x-y|\leq r)} |x-y|^{2mp-|\beta|p-n-a}|b_\beta(y)|^p\,dy = o(1), r\to 0\,,$$

$$\sup_{x\in G} \int_{G\cap(y:|x-y|\leq r)} |x-y|^{2mq-|\beta|q-n-a}|b'_\beta(y)|^q\,dy = o(1), r\to 0\,,$$

Let $(A_p + B)$ and $(A'_q + B')$ be the operators in $L^p(G)$ and $L^q(G)$, respectively, $1 < p < \infty$, $q = p(p-1)^{-1}$, with domains

$$D(A_p + B) = \{u : u \in D(A_p),\ Bu \in L^p(G)\}\,,$$

$$D(A'_q + B') = \{v : v \in D(A'_q),\ B'v \in L^q(G)\}\,.$$

Then:

(a) *$(A_p + B)^* = A'_q + B'$, while the domain of $A_p + B$ equals $D(A_p)$.*

(b) *If $p = q = 2$, then $\mathrm{Re}((A_2 + B)u, u) \geq -K(u, u)$ for some constant K and all u in $D(A_2)$.*

Proof of Theorem 23. We remark first that (a) implies (b), for taking $H = 2^{-1}(A + A')$, $B_1 = 2^{-1}(B + B')$, we know under the hypotheses of (b), if (a) has already been proved, that $H_2 + B_1$ has as its adjoint $H_2 + B_1$, i.e. $H_2 + B_1$ is self-adjoint. If we establish, as we do below, that for large positive ξ, $(H_2 + B_1 + \xi I)^{-1}$ exists on all of $L^2(G)$, then it follows that $H_2 + B_1$ is bounded from below, while $\mathrm{Re}(A_2u + Bu, u) = ((H_2 + B_1)u, u) \geq -K(u, u)$.

To reduce the proof of (a) to Theorem 22, we must verify under the hypotheses of Theorem 23 that

$$\|b_\beta(x)D^\beta u\|_{L^p} \leq \varepsilon\|u\|^p_{2m,p} + K(\varepsilon)\|u\|^p_{0,p}\,. \tag{5.26}$$

By Proposition 1, it follows from (5.26) as in the proof of Theorem 21 that

$$\|b_\beta(x)D^\beta u\|_{0,p} \leq \varepsilon\|(A_p + \xi I)u\|_{0,p}\,, \tag{5.27}$$

for all sufficiently large positive ξ. It follows by summing over β that by choosing ε sufficiently small (and hence ξ sufficiently large),

$$\|Bu\|_{0,p} \leq \tfrac{1}{2}\|(A_p + \xi I)u\|_{0,p}\,. \tag{5.28}$$

In particular, the hypotheses (a), (b), and (d) of Theorem 22 are verified if we let $A = A_p$, $V = B$, $V' = B'$, and $X = L^p(G)$, as follows from (5.28) and the corresponding inequality

$$\|B'v\|_{0,q} \leq 2^{-1}\|(A'_q + B')v\|_{0,q}\,. \tag{5.29}$$

The verification of the hypothesis (c) of Theorem 22 for this case from (5.28) and (5.29) is equally straightforward. Indeed by its definition every function u from $D(A_p)$ lies in $D(B)$, and the mapping $B(A_p + \xi I)^{-1}$, which is everywhere defined on $L^p(G)$, is bounded and hence closed by (5.28). Similarly $B'(A'_q + \xi I)^{-1}$ is everywhere defined and bounded on $L^q(G)$ by (5.29). Thus, since all the hypotheses of Theorem 22 are verified, the conclusion (a) of Theorem 23 follows from (5.26), and the corresponding inequality for the functions $b'_\beta(x)$. It follows from Theorem 22, in addition, that $(A_p + V + \xi I)^{-1}$ and $(A'_q + V' + \xi I)^{-1}$ exist for large positive values of ξ.

The inequality (5.26), (as well as the corresponding inequality for the b'_β) follows from the simpler inequality

$$
\begin{aligned}
|D^\beta u(x)|^p \leqq K(a) r^a \int\limits_{G \cap (|x-y| \leqq r)} |x-y|^{2mp-|\beta|p-n-a} \sum_{|\alpha|=2m} |D^\alpha u(y)|^p \, dy + \\
+ K(a) r^{a-p(2m-|\beta|)} \int\limits_{G \cap (|x-y| \leqq r)} |x-y|^{2mp-|\beta|p-n-a} |u(y)|^p \, dy ,
\end{aligned} \tag{5.30}
$$

by multiplying by $|b_\beta(x)|^p$, integrating over G, and choosing r sufficiently small. Finally, (5.30) follows from (5.11) by replacing $u(x)$ by $D^\beta u(x)$ and applying (5.7) and (5.8) as in the proof of Lemma 16.

In the remainder of the discussion of this section, we shall restrict ourselves to the case of $L^2(G)$.

Theorem 24. *Let A be a regularly elliptic, uniformly strongly elliptic differential operator of order $2m$ defined on an open subset G of E^n, with G uniformly regular of class C^{2m} and locally regular of class C^{4m}. Suppose that $V(x)$ is a regular potential in the sense of Definition 8, i.e. V lies locally in L^r for $r > \min(2, n(2m)^{-1})$ and $\mathrm{Re}(V(x)) \geqq -c$ for all x in G. Let $(A_2 + V(x)I)^\#$ and $(A'_2 + \overline{V}(x)I)^\#$, have as their domain $D(A_2) \cap R(G)$. Let $(A_2 + V(x)I)$ and $(A'_2 + \overline{V}(x)I)$ be the closures of $(A_2 + V(x)I)^\#$ and $(A'_2 + \overline{V}(x)I)^\#$, (the closures existing since each of the latter operators has a densely defined adjoint, namely an extension of the other).*

Then:

(i) $(A_2 + V(x)I)^* = (A'_2 + \overline{V}(x)I)$.

(ii) *There exist constants $c_0 > 0$, $c \geqq 0$, such that for all u in $D(A_2 + V(x)I)$, $u \in W^{m,2}(G)$ and*

$$\mathrm{Re}((A_2 + V(x)I)u, u) \geqq c_0 \|u\|^2_{m,2} - c\|u\|^2_{0,2} .$$

(iii) *If u is any function in $D(A_2 + V(x)I)$ and N is a bounded open subset of G, then u lies in $W^{2m,2}(N)$. On the boundary of G, u satisfies the null Dirichlet boundary conditions of order m. If $(A + V(x)I)u \in R(G)$, then u lies in $W^{2m,p}(N)$ for every bounded open subset N of G and any $p < \infty$.*

(iv) *If A is formally self-adjoint and $V(x)$ is real valued, then $(A_2 + V(x)I)$ is essentially self-adjoint.*

(v) *$(A + V(x)I + \xi I)^{-1}$ exists on all of $L^2(G)$ for ξ large and positive.*

Before proceeding to the proof of Theorem 24, we must establish some more detailed results on differential operators A which satisfy the regularity condition (IV). These results are of significance in their own right, and will be applied in the later discussion.

Definition 10. *Let B be a differential operator on the open subset G of E^n. Then B is said to be a-regular if there exists a constant $\delta > 0$ and an open subset G_1 of E^n with $d(G, E^n - G_1) > \delta$ such that both B and its formal adjoint B' have uniformly bounded coefficients on G_1.*

In terms of Definition 10, the regularity condition (IV) on our differential operator A asks for the a-regularity of A together with the uniform continuity of the top-order coefficients of A on some domain G_1 containing G as in Definition 10.

Theorem 25. *Let G be an open subset of E^n with G uniformly regular of class C^{2m} and locally regular of class C^{4m}. Let B be an a-regular differential operator of order k, $k \leqq 2m$, on G. Then:*

(a) *If $k = 2m$, there exists a constant $K > 0$ such that*

$$|(Bu, v)| \leqq K\|u\|_{m,2}\|v\|_{m,2}$$

for u and v in $D(A_2)$.

(b) *If $k = 2m_1$, $m_1 < m$, then there exists a constant $K > 0$ such that*

$$|(Bu, v)| \leqq K\|u\|_{m_1,2} \cdot \|v\|_{m_1,2},$$

for all u, v in $D(A_2)$. More generally if $k = m_2 + m_3$, $m_2 \leqq m$, $m_3 \leqq m$, we have an inequality of the type

$$|(Bu, v)|^2 \leqq K\{\|u\|^2_{m_2,2} + \|v\|^2_{m_2,2}\} \cdot \{\|u\|^2_{m_3,2} + \|v\|^2_{m_3,2}\},$$

for all u and v in $D(A_2)$.

(c) *If order $B < 2m$, there exists a constant $K > 0$ such that if λ is any real-valued function from $R(G)$, then for $u, v \in D(A_2)$, k = order of B,*

$$|(B(\lambda^{2m}u), v) - (\lambda^m B(\lambda^m u), v)| \leqq$$

$$\leqq K R_k \sum_{|\alpha|,|\beta| \leqq m;\ |\alpha|+|\beta|<2m} \|\lambda^{|\alpha|} D^\alpha u\|_{0,2}\|\lambda^{|\beta|} D^\beta v\|_{0,2},$$

with

$$R_k = \{1 + \sup|\lambda(x)|\}^{2m-k}\left\{\sum_{0<|\alpha|\leqq 2m} \sup|D^\alpha\lambda(x)|\right\}\left(1 + \sum_{0<|\alpha|\leqq 2m} \sup|D^\alpha\lambda|\right)^{k-1}.$$

(d) *If order $(B) = 2m$*

$$|\mathrm{Re}\{(B(\lambda^{2m}u), u) - (\lambda^m B(\lambda^m u), u)\}| \leqq$$

$$\leqq K R_{2m}\left\{\sum_{|\alpha|\leqq m}\|\lambda^{|\alpha|}D^\alpha u\|_{0,2}\right\}\left\{\sum_{|\alpha|\leqq m-1}\|\lambda^{|\alpha|}D^\alpha u\|\right\}_{0,2}.$$

Remark. Applying Theorem 25 to the a-regular differential operator A which is uniformly strongly elliptic and satisfies the regularity condition (IV), we see as a consequence of Theorem 16 as well as the present theorem that the sesqui-linear form (Au, v) defined on $D(A_2)$ satisfies the inequalities

$$c_0\|u\|^2_{m,2} - K\|u\|^2_{0,2} \leqq \mathrm{Re}(Au, u)\ |(Au, v)| \leqq K\|u\|_{m,2} \cdot \|v\|_{m,2}$$

and therefore gives rise to a variational boundary-value problem of elliptic type, in a by now well-known sense. We may extend forms like those described in Theorem 25 by continuity to all of $W_0^{m,2}(G)$, the closure of $D(A_2)$ in $W^{m,2}(G)$.

Proof of Theorem 25. We begin with the incidental remark that if B is given in the generalized divergence form $\sum D^\beta(a_{\alpha\beta}(X)D^\alpha)$, with uniformly

bounded coefficients $a_{\alpha\beta}(x)$ on G and suitable restrictions on $|\alpha|$ and $|\beta|$ in the summation, the results of Theorem 25 would follow from integration by parts in the usual fashion. In the case $m = 1$, our hypotheses imply that B can be written in such a form, as we shall point out in detail in Section 7, but for $m > 1$, this fact is not so obvious. (It would be rather interesting to know whether every a-regular operator B can be written in a suitable divergence form, and in particular whether every operator A of order $2m$ satisfying the regularity conditions (IV) can be put in the form $Au = \sum_{|\alpha|,|\beta| \leqq m} D^\beta \cdot (a_{\alpha\beta}(x) D^\alpha u)$ with uniformly bounded coefficients $a_{\alpha\beta}$. We shall establish Theorem 25 without proving such a result, but the existence of Theorem 25 under our given hypotheses might seem to indicate the likelihood that A can be put in such a generalized divergence form.)

Proof of Theorem 25(a). Let $\triangle$ be the ordinary Laplace operator, and for large positive K, consider the operator $K(-\triangle)^m - B = B_K$. For large enough K, B_K will be uniformly strongly elliptic and regularly elliptic on G, $B'_K = K(-\triangle)^m - B'$, and B_K obviously satisfies the regularity condition (IV). Therefore by Theorem 16, for a suitable constant K_1

$$0 \leqq \mathrm{Re}(B_K u, u) + K_1(u, u) = K((-\triangle)^m u, u) - \mathrm{Re}(Bu, u) + K_1(u, u) .$$

Putting the term involving B on the left we obtain

$$\mathrm{Re}(Bu, u) \leqq K_2 \|u\|^2_{m,2}$$

for u in $D(A_2)$. We apply this conclusion to the two differential operators B_1 and B_2 formed from B by $B_1 = 2^{-1}(B + B')$, $B_2 = (2i)^{-1}(B - B')$. Both B_1 and B_2 are formally self-adjoint, while $B = B_1 + iB_2$. Further, B_1 and B_2 are both a-regular if B is a-regular. Hence we have

$$\mathrm{Re}(B_1 u, u) = (B_1 u, u) \leqq K_2 \|u\|^2_{m,2} ,$$

$$\mathrm{Re}(B_2 u, u) = (B_2 u, u) \leqq K_2 \|u\|^2_{m,2} ,$$

for u in $D(A_2)$. Applying the same argument to $(-B_1)$ and $(-B_2)$, we obtain

$$|(B_1 u, u)| \leqq K_2 \|u\|^2_{m,2} , \quad |(B_2 u, u)| \leqq K_2 \|u\|^2_{m,2} .$$

We now polarize each of the self-adjoint operators B_1 and B_2. Carrying through the argument in detail for B_1, we have

$$4\,\mathrm{Re}(B_1 u, v) = (B_1(u + v), u + v) - (B_1(u - v), u - v) ,$$

$$4|\mathrm{Re}(B_1 u, v)| \leqq K_2\{\|u + v\|^2_{m,2} + \|u - v\|^2_{m,2}\} \leqq 4K_2(\|u\|^2_{m,2} + \|v\|^2_{m,2})$$

and replacing u by ζu, where ζ is the complex number of modulus one defined as $\overline{(B_1 u, v)} \cdot |(B_1 u, v)|^{-1}$, we have

$$|(B_1 u, v)| \leqq 4K_2\{\|u\|^2_{m,2} + \|v\|^2_{m,2}\} .$$

In particular, since the sup of $|(B_1 u, v)|$ for u and v of $W^{m,2}$-norm one is less then $8K_2$, we have $|(B_1 u, v)| \leqq 8K_2 \|u\|_{m,2} \cdot \|v\|_{m,2}$. Similarly, $|(B_2 u, v)| \leqq$ $\leqq 8K_2 \|u\|_{m,2} \cdot \|v\|_{m,2}$, since $B = B_1 + iB_2$,

$$|(Bu, v)| \leqq |(B_1 u, v)| + |(B_2 u, v)| \leqq 16K_2 \|u\|_{m,2} \cdot \|v\|_{m,2} ,$$

and Theorem 25(a) is proved.

Proof of Theorem 25(b). The first assertion of Theorem 25(b) follows from Theorem 25(a), since all the hypotheses holding for $2m$ imply that they hold for $2m_1$, $m_1 < m$.

Suppose now that k is even, $k = 2m_1$. Then for $m_2 = m_1 + r$, $m_3 = m_1 - r$, it follows from Lemma 6 that

$$|(Bu, u)| \leqq K\|u\|^2_{m_1,2} \leqq K_3\|u\|_{m_2,2}\|u\|_{m_3,2} \leqq K_3\varepsilon\|u\|^2_{m_2,2} + K_3\varepsilon^{-1}\|u\|^2_{m_3,2}\,.$$

Applying the last inequality to B_1 and B_2, as in the proof of (a), and polarizing, we obtain

$$|(Bu, v)| \leqq K_3\varepsilon\{\|u\|^2_{m_2,2} + \|v\|^2_{m_2,2}\} + K\varepsilon^{-1}\{\|u\|^2_{m_3,2} + \|v\|^2_{m_3,2}\}\,.$$

Setting

$$\varepsilon = (\|u\|^2_{m_2,2} + \|v\|^2_{m_2,2})^{1/2}(\|u\|^2_{m_3,2} + \|v\|^2_{m_3,2})^{-1/2},$$

we obtain the inequality of Theorem 5(b) for this case.

To obtain the corresponding inequality for the case of k odd, $k = 2m_1 + 1$, it is sufficient to establish the special case

$$|(Bu, u)| \leqq K\|u\|_{m_1+1,2}\|u\|_{m_1,1}\,, \tag{5.31}$$

for by Lemma 6 of Section 1, for $m_2 + m_3 = k$, $m_2 = m_1 + r$, $m_3 = m + 1 - r$,

$$\|u\|_{m_1,2} \leqq K\{\|u\|_{m_2,2}\}^{r(2r+1)^{-1}}\{\|u\|_{m_3,2}\}^{(r+1)(2r+1)^{-1}},$$

$$\|u\|_{m_1+1,2} \leqq K\{\|u\|_{m_2,2}\}^{(r+1)(2r+1)^{-1}}\{\|u\|_{m_3,2}\}^{r(2r+1)^{-1}},$$

which together imply that $|(Bu, u)| \leqq K\|u\|_{m_2,2}\|u\|_{m_3,2}$, and the remainder of the proof is identical with the case of even k. The inequality (5.31) follows by the arguments applied above from the simpler inequality

$$\mathrm{Re}(Bu, u) \leqq \varepsilon\|u\|^2_{m_1+1,2} + K\varepsilon^{-1}\|u\|^2_{m_1,2}\,, \quad (0 < \varepsilon < 1) \tag{5.32}$$

or, letting B_1 be the symmetric part of B as above, from the inequality

$$+ \big((\varepsilon(-\triangle)^{m_1+1} + K\varepsilon^{-1}(-\triangle + 1)^{m_1} - B_1)u, u\big) \geqq 0 \tag{5.33}$$

for all u in $D(A_2)$ and a fixed constant $K > 0$ independent of ε or u. Since the differential operator $C_K = \varepsilon(-\triangle)^{m_1+1} + K\varepsilon^{-1}(-\triangle + 1)^{m_1} - B_1$ is formally self-adjoint and satisfies the regularity condition (IV), its realization in $L^2(G)$ is self-adjoint by Theorem 16. Hence to prove the inequality (5.33), i.e. to show that C_K is non-negative, it suffices to show that $\|(C_K + \xi I)u\| \geqq \|u\|$ for $\xi \geqq 0$, u in $D(A_2)$, norms without other designation being taken in $L^2(G)$. To prove this last fact, we consider

$$\begin{aligned}\|(C_K + \xi I)u\|^2 = {} & \varepsilon^2\|\triangle^{m_1+1}u\|^2 + K^2\varepsilon^{-2}\|(-\triangle + 1)^{m_1}u\|^2 + \\ & + \|B_1u\|^2 + \xi^2\|u\|^2 + 2K((-\triangle)^{m_1+1}u, (-\triangle + 1)^{m_1}u) + \\ & + 2(-1)^{m_1}\varepsilon\,\mathrm{Re}(\triangle^{m_1+1}u, B_1u) - 2K\varepsilon^{-1}((-\triangle + 1)^{m_1}u, B_1u) + \\ & + 2\varepsilon\xi((-\triangle)^{m_1+1}u, u) + 2K\varepsilon^{-1}\xi((-\triangle + 1)^{m_1}u, u) - 2\xi\,\mathrm{Re}(B_1u, u)\,.\end{aligned} \tag{5.34}$$

We remark that for $u \in D(A_2)$,

$$\|B_1 u\|^2 \geqq 0\,, \tag{5.35}$$

$$\varepsilon^2 \|\triangle^{m_1+1} u\|^2 \geqq \varepsilon^2 \|u\|^2_{2m_1+2,2} - \varepsilon^2 \|u\|^2_{2m+1,2}\,, \tag{5.36}$$

$$\|(-\triangle + 1)^{m_1} u\|^2 \geqq C_0 \|u\|^2_{2m_1,2}\,, \quad (C_0 > 0)\,, \tag{5.37}$$

$$((-\triangle)^{m_1+1} u, (-\triangle + 1)^{m_1} u) \geqq C_0 \|u\|^2_{2m_1+1,2} - C_1 \|u\|^2_{2m_1,2}\,, \tag{5.38}$$

$$((-\triangle)^{m_1+1} u, u) \geqq 0\,, \tag{5.39}$$

$$((-\triangle + 1)^{m_1} u, u) \geqq 0\,. \tag{5.40}$$

Combining (5.34) with (5.35)—(5.40), we obtain

$$\begin{aligned} \|(C_K + \xi I) u\|^2 \geqq{}& \varepsilon^2 \|u\|^2_{2m+2,2} - \varepsilon^2 \|u\|^2_{2m+1,2} + \\ &+ C_0 K^2 \varepsilon^{-2} \|u\|^2_{2m_1,2} + \xi^2 \|u\|^2 + 2K C_0 \|u\|^2_{2m_1+1,2} - 2K C_1 \|u\|^2_{2m_1,2} - \\ &- 2\varepsilon |(\triangle^{m_1+1} u, B_1 u)| - 2K\varepsilon^{-1} |((-\triangle + 1)^{m_1} u, B_1 u)| - 2\xi |(B_1 u, u)|. \end{aligned} \tag{5.41}$$

Next we note that

$$2\xi |(B_1 u, u)| \leqq \frac{\xi^2 \|u\|^2}{2} + 2\|B_1 u\|^2 \leqq \frac{\xi^2 \|u\|^2}{2} + K_4 \|u\|^2_{2m+1,2}\,. \tag{5.42}$$

Combining the separate terms in (5.41) containing the same norm and choosing $K \geqq K_0$, (K_0 large but independent of ε or u), we have

$$\begin{aligned} \|(C_K + \xi I) u\|^2 \geqq{}& \varepsilon^2 \|u\|^2_{2m,2} + C_0^1 K^2 \varepsilon^{-2} \|u\|^2_{2m_1,2} + K C_0 \|u\|^2_{2m+1,2} - \\ &- 2\varepsilon |(\triangle^{m_1+1} u, B_1 u)| - 2K\varepsilon^{-1} |(-\triangle + 1)^{m_1} u, B_1 u)|\,. \end{aligned} \tag{5.43}$$

However,

$$2\varepsilon |(\triangle^{m_1+1} u, B_1 u)| \leqq 2^{-1} \varepsilon^2 \|u\|^2_{2m_1+2,2} + K_1 \|u\|^2_{2m_1+1,2}\,, \tag{5.44}$$

$$2K\varepsilon^{-1} |(-\triangle + 1)^{m_1} u, B_1 u)| \leqq 2^{-1} C_0 K^2 \varepsilon^{-2} \|u\|^2_{2m_1,2} + K_2 \|u\|^2_{2m_1+1,2}\,, \tag{5.45}$$

with K_1 and K_2 constants independent of ε. Introducing (5.44) and (5.45) into (5.43), we obtain

$$\|(C_k + \xi I) u\|^2 \geqq (K c_0 - K_1 - K_2) \|u\|^2_{2m_1+1,2} \geqq \|u\|^2\,, \tag{5.46}$$

if $K > c_0^{-1}(K_1 + K_2 + 1)$. Thus (5.31) is proved, and with it Theorem 25(b).

Proof of Theorem 25(c). The differential operator $Cv = B(\lambda^{2m} v) - \lambda^m B(\lambda^m v)$ is of order $< 2m$, while C', its formal adjoint, is given by $C'v = \lambda_{2m} B' v - \lambda^m B'(\lambda^m v)$. It follows immediately from our hypotheses that for any fixed function λ, the operator C (which depends implicitly upon λ) is a-regular. As a consequence, if we apply Theorem 25(b) to the operator C, we obtain the identity

$$|(Cu, v)| \leqq K(\lambda) \{\|u\|_{m,2} \|v\|_{m-1,2} + \|u\|_{m-1,2} \|v\|_{m,2}\}\,. \tag{5.47}$$

To establish the conclusion of Theorem 24(c), it is necessary to sharpen the inequality (5.47) by establishing the form of the constant $K(\lambda)$ and by replacing the norms of u and v by sums of corresponding norms of $\lambda^j u$ and $\lambda^j v$, respectively. Replacing B by the two symmetric operators B_1 and B_2, we may assume without loss of generality that B is formally self-adjoint. In particular, since

the order of B is $\leq 2m$, the coefficients of $D^\beta u$ in B with $|\beta| = 2m$, if any such terms exist, may be taken to be real-valued functions.

Suppose the order of B is $< 2m$. Let $B = \sum_{|\beta| \leq 2m-1} b_\beta(x) D^\beta$, $B' = B$. Consider the corresponding operator C defined in the preceding paragraph. Then

$$(5.48) \qquad Cv = B(\lambda^m \lambda^m v) - \lambda^m B(\lambda^m v) = \sum_{0 < |\beta| \leq 2m-1} D^\beta(\lambda^m)\, C_\beta(\lambda^m v)\,,$$

with C_β a differential operator of order $\leq 2m - 1 - |\beta|$ with coefficients which do not depend on λ.

Let E be any a-regular differential operator of order s, ϕ any function in $C^\infty(E^n)$. If $E = E^{(1)} + iE^{(2)}$, $E^{(1)}$ and $E^{(2)}$ symmetric and a-regular, then

$$(5.49) \qquad E(\phi w) - \phi E(w) = \sum_{0 < |\beta| \leq s} D^\beta(\phi)\, E_\beta(w)\,,$$

with E_β a differential operator of order $\leq s - |\beta|$ which may be written as $E_\beta^{(1)} + iE_\beta^{(2)}$ where $E_\beta^{(1)}$ and $E_\beta^{(2)}$ are the corresponding operators in the analogous decompositions for $E^{(1)}$ and $E^{(2)}$ respectively. If we wish to prove that E_β is itself a-regular, it suffices to establish this for $E_\beta^{(1)}$ and $E_\beta^{(2)}$, i.e. to restrict ourselves to the case in which E is formally self-adjoint. In any case, (5.49) is an identity for ϕ and w in $C^\infty(E^n)$. Taking the adjoints of the differential operators on both sides (obtaining operators with distribution coefficients on the right, as far as we know until the present proof is complete), we obtain

$$(5.50) \qquad \phi E(w) - E(\phi w) = \sum E'_\beta(D^\beta(\beta) w) \quad \text{(summation for } 1 \leq |\beta| \leq s)\,,$$

But the left-hand side is the negative of the left side of (5.49), so that

$$(5.51) \qquad -\sum_{1 \leq |\beta| \leq s} D^\beta(\phi)\, E_\beta(w) = \sum_{1 \leq |\beta| \leq s} E'_\beta(D^\beta(\phi) w)\,.$$

We now set $\phi_\beta = (\beta!)^{-1} x^\beta$, where $\beta! = \prod_j (\beta_j)!$, into equation (5.51). We obtain

$$(5.52) \qquad -E_\beta(w) - \sum_{1 \leq |\alpha| < |\beta|} D^\alpha(\phi_\beta)\, E_\alpha(w) = E'_\beta(w) + \sum_{1 \leq |\alpha| < |\beta|} E'_\alpha(D^\alpha(\phi_\beta) w)\,.$$

For $|\beta| = 1$, we obtain $E'_\beta(w) = -E_\beta(w)$. For $|\beta| > 1$, we obtain by recursion that if E'_α for $|\alpha| < |\beta|$ has bounded coefficients on some domain G_1, then so does E'_β. It follows that all the E'_β have bounded coefficients on G_1, and since we know already that the E_β do also, the operators E_β are a-regular for all β.

We now apply the a-regularity of the operators E_β in the decomposition (5.49) to the equation (5.48) with $E = B$ and $w = \lambda^m v$. It follows that the operators C_β in (5.48) are all a-regular. The term T_β of index β in equation (5.48) is of the form $r_\beta(\lambda)\, \lambda^{m-|\beta|} C_\beta(\lambda^m v)$, with $r_\beta(\lambda)$ a sum of products of derivatives of λ of order $\leq |\beta|$. If $|\beta| \geq m$, $|(Tu, v)| \leq \sup |r(\lambda)|\, \|\lambda^m u\|_{m,2} \|v\|$, and by an obvious expansion, this last term is dominated by the right hand side of the inequality of Theorem 25(c). For $|\beta| \leq m - 1$, and thereby the order of C_β possibly greater than m, an additional argument is necessary. We

note that by (5.49)

$$T_\beta u = r_\beta(\lambda)\lambda^{m-|\beta|} C_\beta(\lambda^m v) = r_\beta(\lambda)\,\lambda^{m+1-|\beta|} C_\beta(\lambda^{m-1}u) + \tag{5.53}$$

$$+ \sum_{0<|\alpha|<2m-1-|\beta|} r_\beta(\lambda)\,\lambda^{m-|\beta|} D^\alpha(\lambda)\,E_\alpha(\lambda^{m-1}u)\,,$$

with the order of the a-regular differential operator $E_\alpha \leqq$ order $(C_\beta) - |\alpha|$. Continuing the decomposition initiated in (5.53) in the obvious fashion and transfering powers of λ from the inside of the differentiated parenthesis to the outside (and successively accumulating further lower-order terms of a similar form), we arrive finally at an expression of the form

$$T_\beta u = \sum_{|\beta|=2j,\, j<m} r_\beta(\lambda)\,\lambda^j F_\beta(\lambda^j u) + \tag{5.54}$$

$$+ \sum_{|\beta|=2j+1,\, j<m-1} r_\beta(\lambda)\,\lambda^{j+1} F_\beta(\lambda^{j+1}u) + R\,,$$

where each of the operators F_β is a-regular and of order $|\beta|$, R is a-regular and of order $\leqq m$, and the functions $r_\beta(\lambda)$ are of the form $\lambda^a s(\lambda)$, with $a \geqq 0$ and $s(\lambda)$ a sum of products of derivatives of λ. It follows immediately from Theorem 25(b) that $|(T_\beta u, v)|$ is bounded by the right-hand side of the inequality of Theorem 25(c), so that the proof is complete if the order of B is less than $2m$ and Theorem 25(c) is established.

Proof of Theorem 25(d). For the case in which the order of $B = 2m$, some modifications must be made in the preceding proof. We recall that we may assume that $B = B'$ and that therefore the highest order terms of B have real coefficients. We obtain (5.48) once more, but in this case the operators C_β are in general of order $(2m - |\beta|)$. Applying (5.49) as above and letting $T_\beta u = D^\beta(\lambda^m)\,C_\beta(\lambda^m u)$, we obtain by applying the reasoning previously applied to obtain (5.53) that $T_\beta u$ may be written in the form

$$T_\beta u = \sum_{|\beta|=2j,\, j<m} r_\beta(\lambda)\lambda^j F_\beta(\lambda^j u) + \sum_{|\beta|=2j+1,\, j\leqq m-1} i\,r_\beta(\lambda)\,\lambda^j F_\beta(\lambda^{j+1}u) + R, \tag{5.55}$$

with F_β and R a-regular as before, R of order $\leqq m$, and F_β of order $|\beta|$. The first and third terms are estimated as above. It remains to deal with the second sum on the right.

To clarify the exposition for this case, we give a series of remarks which we shall list in serial order:

(1) In (5.55), we have written the imaginary unit i in front of $r_\beta(\lambda)$ to emphasize that in the crucial case, $r_\beta(\lambda)$ in (5.55) can be chosen to be real-valued while the highest order terms of F_β have real coefficients. Replacing F_β by $2^{-1}(F_\beta + F'_\beta)$ which differs from F_β by an a-regular lower-order operator, we may assume F_β formally self-adjoint.

(2) To estimate $\mathrm{Re}\,(Cu, u)$, it suffices to estimate $\mathrm{Re}\,(T_\beta u, u)$ for various β. By our first remark, since the lower order terms have already been handled in the proof of Theorem 25(c), it suffices to estimate

$$\mathrm{Re}\,(i\,r_\beta(\lambda)\lambda^j F_\beta(\lambda^{j+1}u), u)$$

with $r_\beta(\lambda)$ real-valued, F_β formally self-adjoint and of order $(2j+1)$. We note however that

$$(5.56)\qquad i(r_\beta(\lambda)\lambda^j F_\beta(\lambda^{j+1}u), u) = i(F_\beta(\lambda^{j+1}u), r_\beta(\lambda)\lambda^j u) = i\left(\lambda^j u, \lambda F_\beta(r_\beta(\lambda)\lambda^j u)\right),$$

while

$$(5.57)\qquad \lambda F_\beta(r_\beta(\lambda)\lambda^j u) = \lambda r_{\beta,1}(\lambda) F_{\beta,1}(\lambda^j u) + \sum_{0<|\alpha|} r_\beta(\lambda) D^\alpha(\lambda) F_{\beta,\alpha}(\lambda^j u) + r_\beta(\lambda) F_\beta(\lambda^{j+1}u),$$

where the function $r_{\beta,1}$ is of the same type as r_β, and the operators $F_{\beta,1}$ and $F_{\beta,\alpha}$ are a-regular and of order less than F_β. Denoting the sum of all the terms on the right side of (5.57) except the last term as R_1, we have

$$(5.58)\qquad i(r_\beta(\lambda)j^j F_\beta(\lambda^{j+1}u), u) = i(u, \lambda^j r_\beta(\lambda) F_\beta(\lambda^{j+1}u)) + i(\lambda^j u, R_1 u).$$

But (5.58) asserts that

$$(5.59)\qquad \mathrm{Re}(T_\beta u, u) = (\lambda^j u, R_1 u),$$

while the latter term is one that can be estimated by the argument of the proof of Theorem 25(c). Hence the estimation of $|\mathrm{Re}(Cu, u)|$ follows, and Theorem 25(d) is proved.

Proof of Theorem 24. By Theorem 16, there exists a constant K such that $\mathrm{Re}(A_2 u, u) \geqq -K\|u\|^2_{0,2}$ for all u in $D(A_2)$. Since $V(x)$ lies locally in L^2 on a neighborhood of each point of $G \cup \Gamma$, the operator $(A_2 + V(x)I)$ is well-defined for every function u in $D(A_2) \cap R(G)$, and $(A_2 + V(x)I)u$ lies in $L^2(G)$. Moreover, since $\mathrm{Re}(V(x)) \geqq -c$ for all x in G,

$$(5.60)\qquad \mathrm{Re}\left((A_2 + V(x)I)u, u\right) \geqq -(K+c)\|u\|^2_{0,2}$$

for all u in $D(A_2) \cap R(G)$. It follows that the operator $(A_2 + V(x)I + \xi I)^\#$ is one-to-one for $\xi > (K+c)$ and has a bounded inverse on its range. It follows immediately that $(A_2 + V(x)I + \xi I)^\#$, and hence $(A_2 + V(x)I)^\#$ itself, is closable as an operator in $L^2(G)$. Similarly $(A_2' + \overline{V}(x)I)^\#$ is closable in $L^2(G)$. We shall denote the closures of $(A_2 + V(x)I)^\#$ and $(A_2' + \overline{V}(x)I)^\#$ by $(A_2 + V(x)I)$ and $(A_2' + \overline{V}(x)I)$, respectively.

Let $u \in D\left((A_2 + V(x)I)^\#\right)$, $v \in D\left((A_2' + \overline{V}(x)I)^\#\right)$. Since $(A_2)^* = A_2$, and indeed by a previous more elementary argument,

$$(5.61)\qquad \left((A_2 + V(x)I)u, v\right) = \left(u, (A_2' + \overline{V}(x)I)v\right).$$

By continuity, equation (5.61) goes over to u in $D(A_2 + V(x)I)$ and v in $D(A_2' + \overline{V}(x)I)$. Hence $(A_2' + \overline{V}(x)I) \subseteq (A_2 + V(x)I)^*$. By a similar argument for every real ξ, $(A_2' + \overline{V}(x)I + \xi I) \subseteq (A_2 + V(x)I + \xi I)^*$. But the latter operators are simply $(A_2' + \overline{V}(x)I) + \xi I$ and $(A_2 + V(x)I)^* + \xi I$, so that to prove the equality of $(A_2' + \overline{V}(x)I)$ and $(A_2 + V(x)I)^*$ it suffices to do so with $V(x)$ replaced by $V(x) + \xi$ for any suitably large positive number ξ. On the other hand, it follows from (5.60) that for such ξ

$$(5.62)\qquad |\left((A_2 + V(x)I + \xi I)u, u\right)| \geqq \|u\|^2_{0,2},$$

for all u in $D(A_2 + V(x)I + \xi I)$. Since the operator $(A_2 + V(x)I + \xi I)$ is closed, it follows from (5.62) that its range is closed and that it has a bounded inverse on its range. An identical argument for $(A_2' + \overline{V}(x)I)$ shows that for ξ sufficiently large and positive, $(A_2' + \overline{V}(x)I + \xi I)$ has a closed range and a bounded inverse defined on that range. Since $(A_2' + \overline{V}(x)I + \xi I) \subseteq (A_2 + V(x)I + \xi I)^*$, however, $(A_2' + \overline{V}(x)I + \xi I)^{-1} \subseteq [(A_2 + V(x)I + \xi I)^*]^{-1}$. To prove the equality of the latter two operators and thereby Theorem 24(i), it suffices to show that the range of $(A_2' + \overline{V}(x)I + \xi I)$ is all of $L^2(G)$ for ξ large and positive and the same fact for $(A_2 + V(x)I + \xi I)$.

Before we complete the proof of Theorem 24(i), we need to establish Theorem 24(ii). We have already remarked in (5.60) that for u in $D(A_2) \cap R(G)$,

$$\mathrm{Re}\big((A_2 + V(x)I)u, u\big) \geqq -K_1(u, u) .$$

Replacing the operator A by the differential operator $A - \varepsilon(-1)^m \triangle^m$, ($\triangle$ = the Laplacian) which is uniformly strongly elliptic and satisfies the same conditions as A for small $\varepsilon > 0$, we obtain for u in $D(A_2) \cap R(G)$,

$$\mathrm{Re}(A_2 u + V(x)u, u) \geqq \varepsilon(-1)^m(\triangle^m u, u) - K_1\|u\|_{0,2}^2 \geqq c_0\|u\|_{m,2}^2 - \\ - K_1\|u\|_{0,2}^2,\ c_0 > 0 . \tag{5.63}$$

For an arbitrary u in $D(A_2 + V(x)I)$, there exists a sequence $\{u_k\}$ from $D(A_2) \cap R(G)$ such that $u_k \to u$ in $L^2(G)$, $A_2 u_k + V(x)u_k \to (A_2 + V(x)I)u$ in $L^2(G)$. Applying (5.63) to $(u_k - u_j)$, we obtain

$$\|u_j - u_k\|_{m,2}^2 \leqq c_0^{-1}\|(A_2 + V(x)I)(u_j - u_k)\|_{0,2}\|u_j - u_k\|_{0,2} + \\ + c_0^{-1}K_1\|u_j - u_k\|_{0,0}^2 . \tag{5.64}$$

Thus $\{u_k\}$ is a Cauchy sequence in $W^{m,2}(G)$ which converges to some v in $W^{m,2}(G)$. Since u_k converges to u in $L^2(G)$ and since limits in $L^2(G)$ are unique, $v = u$ and u must lie in $W^{m,2}(G)$. Applying (5.63) to u_k, we obtain

$$\mathrm{Re}\big((A_2 + V(x)I)u_k, u_k\big) \geqq c_0\|u_k\|_{m,2}^2 - K_1\|u_k\|_{0,2}^2 . \tag{5.65}$$

Taking the limit on k, we obtain

$$\mathrm{Re}\big((A_2 + V(x)I)u, u\big) \geqq c_0\|u\|_{m,2}^2 - K_1\|u\|_{0,2}^2 , \tag{5.66}$$

and the proof of Theorem 24(ii) is complete.

To finish the proof of Theorem 24(i), we first apply Theorem 24(ii) to the operator A' and the potential $\overline{V}(x)$, obtaining

$$\mathrm{Re}\big((A_2' + \overline{V}(x)I)v, v\big) \geqq c_0\|v\|_{m,2}^2 - K_1\|v\|_{0,2}^2 . \tag{5.67}$$

Suppose now that the range of the operator $(A_2' + \overline{V}(x)I + \xi I)$ is not dense for some real $\xi > K_1 + 1$. Then there exists w in $L^2(G)$ such that

$$\big((A_2' + \overline{V}(x)I + \xi I)v, w\big) = 0 \tag{5.68}$$

for all v in $D(A_2' + \overline{V}(x)I)$. Let us first restrict ourselves to v in $D(A_2) \cap R(G)$ which have compact support in $G_1 \cup \Gamma_1$, where G_1 is a bounded subdomain of G and Γ_1 is the intersection of Γ with the boundary of G_1. If we restrict A_2' to such v, the resulting operator in $L^2(G_1)$ has $A_{2,0}'$ as its closure, where the

latter operator is defined as in Definition 6 of Section 2 with respect to the portion Γ_1 of the boundary of G_1. If we denote the restriction of w to G_1 by w_1, equation (5.68) implies that

$$((A'_{2,0} + \xi I)v, w_1) = -(v, V(x)w_1)\,, \tag{5.69}$$

for all v in $D(A'_{2,0}) \cap R(G) = D(A'_{p,0}) \cap R(G)$, $1 < p < \infty$. By the hypothesis of Theorem 24, V lies locally in L^r for $r > n(2m)^{-1}$ and hence in $L^r(G_1)$. Since w_1 lies in $L^2(G_1)$, Vw_1 lies in $L^t(G_1)$ where $t > 1$ and $t > 2n(4m + n)^{-1}$. In particular, (5.69) implies that

$$((A'_{t_1,0} + \xi I)v, w_1) = (v, f)\,, \tag{5.70}$$

for all v in $D(A'_{t_1,0}) \cap R(G_1)$, $f \in L^t(G_1)$, and t_1 the conjugate exponent of t. By Theorems 12 and 15, it follows that w_1 lies in $D(A_{t,1}) \subset W^{2m,t}_{\text{loc}}(G_1 \cup \Gamma_1)$. By Lemma 5, w_1 lies in $W^{0,s}_{\text{loc}}(G_1 \cup \Gamma_1)$, with $s^{-1} \geqq t^{-1} - 2mn^{-1} \geqq 2^{-1} - (2mn^{-1} - r^{-1})$. Applying this argument to another bounded domain G_2 containing G_1 in its interior, we find that $w_1 \in L^s(G_1)$. Since $f = V(x)w_1$ now lies in $L^{t_2}(G_1)$ with $t_2^{-1} = s^{-1} + r^{-1} = 2^{-1} + 2r^{-1} - 2mn^{-1}$, it follows as before that w_1 lies in $W^{2m,t_2}_{\text{loc}}(G_1)$, and hence by expanding the domain to G_2, in $L^{t_3}(G_1)$ with $t_3^{-1} \geqq (2^{-1} + 2r^{-1} - 2mn^{-1}) - 2mn^{-1} = 2^{-1} - 2(2mn^{-1} - r^{-1})$. Proceeding step by step, we find that for any j, w_1 lies in $L^{t_j}(G_1)$ with $t_j^{-1} \geqq 2^{-1} - (j-1)(2mn^{-1} - r^{-1})$. Since the term in the last parenthesis is positive and j is unrestricted, it follows finally that w lies in $C(G_1 \cup \Gamma_1)$, and that w lies in $W^{2m,p}_{\text{loc}}(G \cup \Gamma)$ for any p. We remark that the preceding argument applies if w lies in $D(A_2 + V(x)I)$ and if $(A_2 + V(x)i)w$ lies in $L_{\text{loc}}(G \cup \Gamma)$. This proves the second assertion of Theorem 24(iii). In addition, it follows that w satisfies the null Dirichlet conditions on Γ.

Returning to the proof of Theorem 24(i), we know that w lies in $W^{2m,2}_{\text{loc}}(G \cup \Gamma)$ and that (5.68) holds for all v in $D(A'_2 + \overline{V}(x)I)$. We remark that any function v in $D(A'_2)$ with bounded support lies in $D(A'_2 + \overline{V}(x)I)$ since v is the limit in $W^{2m,2}(G)$ of functions v_n in $D(A'_2) \cap R(G)$ with uniformly bounded support by Lemma 10 of Section 2. Let $\lambda(x)$ be a function in $C_c^\infty(E^n)$ with $0 \leqq \lambda(x) \leqq 1$, $\lambda(x) = 1$ for $|x| \leqq 1$, $\lambda(x) = 0$ for $|x| \geqq 2$, and for each $\varepsilon > 0$, set $\lambda_\varepsilon(x) = \lambda(\varepsilon x)$. If we put $w_\varepsilon(x) = w(x)\lambda_\varepsilon^{2m}(x)$, w_ε lies in $D(A'_2)$ and has support in the bounded set $|x| \leqq 2\varepsilon^{-1}$. Setting w_ε for v in equation (5.68), we have

$$(A'_2(\lambda_\varepsilon^{2m}w), w) + (\overline{V}\lambda_\varepsilon^{2m}w, w) + \xi(\lambda_\varepsilon^{2m}w, w) = 0\,. \tag{5.71}$$

Applying Theorem 25 to the a-regular operator A', we have

$$|(A'(\lambda_\varepsilon^{2m}w), w) - (A'(\lambda_\varepsilon^m w), \lambda_\varepsilon^m w)| \leqq \tag{5.72}$$
$$\leqq K\,R(\varepsilon) \sum_{|\alpha|,|\beta| \leqq m,\, |\alpha|+|\beta|<2m} \|\lambda_\varepsilon^{|\alpha|} D^\alpha w\|_{0,2} \cdot \|\lambda_\varepsilon^{|\beta|} D^\beta w\|_{0,2}\,,$$

(Note that the fact that w may not lie in $W^{2m,2}(G)$ a-priori does not affect 5.72 since we may replace w by ϕw, $\phi = 1$ for $|x| \leqq 2\varepsilon^{-1}$ without affecting either side of (5.72).). By Theorem 25(c) and (d), it follows from the form of λ_ε

that $R(\varepsilon) \leqq K_2\varepsilon$. Taking the real part of both sides of (5.71) and applying (5.72) and (5.67), we obtain for $\xi > K_1 + 1$,

$$c_0\|\lambda_\varepsilon^m w\|_{m,2}^2 + \|\lambda_\varepsilon^m w\|_{0,2}^2 \leqq \operatorname{Re}\big((A_2' + \overline{V}(x)I + \xi I)(\lambda_\varepsilon^m w), \lambda_\varepsilon^m w\big) \leqq \tag{5.73}$$
$$\leqq K_3\varepsilon\Big(\sum_{|\alpha|\leqq m}\|\lambda_\varepsilon^{|\alpha|}D^\alpha w\|_{0,2}\Big)\Big(\sum_{|\beta|\leqq m-1}\|\lambda_\varepsilon^{|\beta|}D^\beta w\|_{0,2}\Big)\,.$$

In addition,

$$\|\lambda_\varepsilon^m w\|_{m,2}^2 = \sum_{|\alpha|\leqq m}(\|D^\alpha(\lambda_\varepsilon^m w)\|_{0,2}^2 \geqq \sum_{|\alpha|\leqq m}\|\lambda_\varepsilon^m D^\alpha w\|_{0,2}^2 - \tag{5.74}$$
$$- K_4\varepsilon\Big(\sum_{|\alpha|\leqq m}\|\lambda_\varepsilon^{|\alpha|}D^\alpha w\|_{0,2}\Big)\Big(\sum_{|\beta|\leqq m-1}\|\lambda_\varepsilon^{|\beta|}D^\beta w\|_{0,2}\Big)\,.$$

Substituting (5.74) into (5.73) and absorbing the last terms in (5.74) into the term on the right, we have

$$c_0\Big(\sum_{|\alpha|\leqq m}\|\lambda_\varepsilon^m D^\alpha w\|_{0,2}^2\Big) \leqq K_5\varepsilon\Big(\sum_{|\alpha|\leqq m-1}\|\lambda_\varepsilon^{|\alpha|}D^\alpha w\|_{0,2}^2\Big)\,, \tag{5.75}$$

after estimating the terms involving m-th derivatives on the right by the usual inequality, $ab \leqq c_0 2^{-1}a^2 + c_1 b^2$, and making $K_5\varepsilon < 1$ by choosing ε sufficiently small. (All our constants K are, of course, independent of both w and ε.)

To estimate the right-hand side of (5.75) in terms of the left, let $D^\alpha = D_j D^\beta$, $|\alpha| < m$. Then

$$\|\lambda_\varepsilon^{|\alpha|}D^\alpha w\|_{0,2}^2 = (\lambda_\varepsilon^{2|\alpha|}D^\alpha w, D_j D^\beta w) = (D_j[\lambda_\varepsilon^{2|\alpha|}D^\alpha w)], D^\beta w) \leqq \tag{5.76}$$
$$\leqq (\lambda_\varepsilon^{2|\alpha|}D_j D^\alpha w, D^\beta w) + K\varepsilon\|\lambda_\varepsilon^{|\alpha|}D^\alpha w\|_{0,2}\|\lambda_\varepsilon^{|\beta|}D^\beta w\|_{0,2}\,.$$

By the usual estimation,

$$\|\lambda_\varepsilon^{|\alpha|}D^\alpha w\|_{0,2}^2 \leqq \|\lambda_\varepsilon^{|\alpha|+1}D_j D^\alpha w\|_{0,2}\|\lambda_\varepsilon^{|\beta|}D^\beta w\| + (K\varepsilon)^2\|\lambda_\varepsilon^{|\beta|}D^\beta w\|_{0,2}^2\,. \tag{5.77}$$

Iterating (5.77) in the usual fashion, we find that

$$\|\lambda^{|\alpha|}D^\alpha w\|_{0,2}^2 \leqq K\sum_{|\beta|=m}\|\lambda_\varepsilon^m D^\beta w\|_{0,2}^2 + \|w\|_{0,2}^2\,. \tag{5.78}$$

Substituting (5.78) in (5.75), we obtain

$$c_0\sum_{|\alpha|=m}\|\lambda_\varepsilon^m D^\alpha w\|_{0,2}^2 + c_0\|\lambda_\varepsilon^m w\|_{0,2}^2 \leqq K_6\varepsilon\Big\{\sum_{|\alpha|=m}\|\lambda_\varepsilon^m D^\alpha w\|_{0,2}^2 + \|w\|^2\Big\}\,. \tag{5.79}$$

Taking ε so small that $K_6\varepsilon < 1$, we have

$$c_0\|\lambda_\varepsilon^m w\|_{0,2}^2 \leqq K\varepsilon\|w\|_{0,2}^2\,, \tag{5.80}$$

which implies that

$$\int_{G\cap\{|x|\leqq\varepsilon^{-1}\}}|w(x)|^2\,dx \leqq K_7\varepsilon\|w\|_{0,2}^2 = K_8\varepsilon\,, \tag{5.81}$$

since w by hypothesis lie in $L^2(G)$. Letting $\varepsilon \to 0$, since the left hand side increases with ε decreasing while the right hand side approaches zero, we have $w = 0$. This proves that the range of $(A_2' + \overline{V}(x)I + \xi I)$ is dense, and the proof of Theorem 24(i) is complete.

Theorem 24(iv) is a special case of Theorem 24(i). To complete the proof of Theorem 24(iii), suppose that $u \in D(A_2 + V(x)I)$ so that $(A_2 + V(x)I)u = f$, f in $L^2(G)$. Passing to bounded subdomains G_1 as in the proof of Theorem 24(i),

we see that u is a solution of $A_{t,1}u = f - V(x)u$. If u lies in L^2, since $V(x)$ lies in $L^r(G_1)$ for $r > \min(2, n(2m)^{-1})$, we have $f - Vu \in L^t(G)$ with $t^{-1} = 2^{-1} + r^{-1}$. Hence u lies in $W^{2m,t}_{\text{loc}}(G_1 \cup \Gamma_1)$, and passing to a larger domain G_2, u lies in $L^{t_1}(G_1)$ with $t^{-1} \geqq t^{-1} - 2mn^{-1} = 2^{-1} - (2mn^{-1} - r^{-1})$. Hence Vu lies in $L^{t_2}(G_1)$ with $t_2^{-1} = 2^{-1} + r^{-1} - (2mn^{-1} - r^{-1})$, and proceeding step by step as before, we establish that Vu lies in $L^2(G_1)$. Hence $f - Vu$ lies in $L^2(G_1)$ u lies in $W^{2m,2}_{\text{loc}}(G_1 \cup \Gamma_1)$, and passing to a larger domain G_2, u lies in $W^{2m,2}(G_1)$. It follows also since u lies in $D(A_{2,1})$ that u satisfies the null Dirichlet boundary conditions of order m on Γ_1, and since Γ_1 is an arbitrary bounded subset of Γ, on all of Γ. Thus the proof Theorem 24(iii) is complete. Theorem 24(v) is an easy consequence of (i) and (ii). Thus the proof of Theorem 24 is complete.

Up to this point we have been considering potentials V which either satisfy a weak integral boundedness condition as in Definition 8 or whose real parts are bounded from below as in Definition 8a. We now turn to a special sub-class of the latter for which we may obtain a generalization of Theorem 17, i.e. whose resolvents are compact.

Theorem 26. *Let A be a regularly elliptic, uniformly strongly elliptic operator of order $2m$ defined on an open subset G of E^n, with G uniformly regular of class C^{2m} and locally regular of class C^{4m}. Let $V(x)$ be a regular potential on G in the sense of Definition* 8a, *and suppose that* $\operatorname{Re}(V(x)) \to +\infty$ *as* $x \to \infty$, *i.e. for each positive integer N, there exists a bounded set G_N in G with* $\operatorname{Re}(V(x)) \geqq N$ *for x in $G - G_N$. Then $(A_2 + V(x)I + \xi I)^{-1}$, which exists on all of $L^2(G)$ for ξ large and positive, is a compact linear transformation of $L^2(G)$, the spectrum of $(A_2 + V(x)I)$ is point spectrum and discrete and its eigenspaces are of finite dimension, while for each ζ in the resolvent set of $(A_2 + V(x)I)$, $(A_2 + V(x)I - \zeta I)^{-1}$ is a compact operator.*

Proof of Theorem 26. It suffices by standard arguments as noted in the proof of Theorem 17 to show that $(A_2 + V(x)I + \xi I)^{-1}$ exists for large positive ξ and is a compact operator in $L^2(G)$. The fact that $(A_2 + V(x)I + \xi I)^{-1}$ exists for ξ large and positive was noted under weaker hypotheses in Theorem 24. We need therefore only show that this operator is compact.

Let $\{u_k\}$ be a weakly convergent sequence in $L^2(G)$, $v_k = (A_2 + V(x)I + \xi I)^{-1}u_k$. By Theorem 24(iii), the restriction of any v in $D(A_2 + V(x)I)$ to a bounded domain G_1 lies in $W^{2m,2}(G_1)$. Since the restriction mapping is obviously a closed mapping of the graph of $(A_2 + V(x)I)$ into $W^{2m,2}(G_1)$, it follows from the closed graph theorem that there must exist a constant $c(G_1)$ so that so all v in $D(A_2 + V(x)I)$,

$$\|v\|_{W^{2m,2}(G_1)} \leqq C(G_1)\{\|v\|_{0,2} + \|(A_2 + V(x)I)v\|_{0,2}\}.$$

Since the sequence $\{u_k\}$ is weakly convergent in $L^2(G)$ and $(A_2 + V(x)I + \xi I)^{-1}$ is strongly continuous and hence weakly continuous, v_k converges weakly to some element v in $L^2(G)$. If u is the weak limit of the u_k, $[u_k, v_k]$ converges weakly to $[u, v]$ in $L^2(G) \times L^2(G)$ while all the terms $[u_k, v_k]$ lie in the graph of $(A_2 + V(x)I + \xi I)^{-1}$. Since the graph is closed and hence weakly closed, v lies in the domain of $(A_2 + V(x)I + \xi I)$ and $(A_2 + V(x)I + \xi I)v = u$.

Since the restriction mapping of $D(A_2 + V(x)I)$ into $W^{2m,p}(G_1)$ is continuous, it is weakly continuous. It follows that v_k converges weakly to v in $W^{2m,2}(G_1)$ and therefore by Lemma 5 strongly in $L^2(G_1)$ since G_1 is bounded. Since

$$\|v - v_k\|_{0,2} \leqq \|v - v_k\|_{L^2(G_1)} + \|v\|_{L^2(G-G_1)} + \|v_k\|_{L^2(G-G_1)},$$

it suffices, in order to show that v_k converges strongly to v, to prove that given $\varepsilon > 0$, there exists a bounded set G_ε such that

$$\|v_k\|_{L^2(G-G_\varepsilon)} \leqq \varepsilon$$

for all k. We know, however, that for w in $D(A_2 + V(x)I) \cap R(G)$,

$$\text{Re}\big((A_2 + V(x)I)w, w\big) \geqq \text{Re}\big(V(x)\big)(w, w) - K\|w\|_{0,2}^2 . \tag{5.82}$$

For an arbitrary element w of $D(A_2 + V(x)I)$, if we take a sequence w_n from $D(A_2 + V(x)I) \cap R(G)$ such that $w_n \to w$ in $L^2(G)$, $(A_2 + V(x)I)w_n \to (A_2 + V(x)I)w$ in $L^2(G)$, it follows from (5.82) that

$$\int_{|x| \leqq N} \text{Re}\{V(x)\}\,|w(x)|^2\,dx = \lim_k \int_{|x| \leqq N} \text{Re}\{V(x)\}\,|w_k(x)|^2\,dx \leqq \tag{5.83}$$

$$\leqq \sup_k (\text{Re}\,(V(x)w_k, w_k) \leqq K .$$

Since the constant K in (5.83) is independent of N, it follows that $(\text{Re}\{(V(x)\}w, w)$ is finite for all w in $D(A_2 + V(x)I)$ with

$$(\text{Re}\,[V(x)]w, w) \leqq K\|w\|_{0,2}^2 + \|w\|_{0,2} \cdot \|(A_2 + V(x)I)w\|_{0,2} . \tag{5.84}$$

In particular if we consider the set G_N such that $\text{Re}(V(x)) \geqq N$ on $G - G_N$ and set $w = v_k$ in (5.84), we obtain

$$N \int_{G-G_N} |v_k(x)|^2\,dx \leqq K_9 , \tag{5.85}$$

with a constant K_9 independent of k or N. Taking N sufficiently large so that $K_9 N^{-1} < \varepsilon^2$, we choose the corresponding G_N to be G_ε, the uniform norm inequality is fulfilled, and Theorem 26 is proved.

Theorem 27. *Let A and G satisfy the conditions of Theorem 26, and suppose in addition that $2m > n$, and that for some positive constants ξ_0 and h, if $k(x) = \inf\{\text{Re}(V(y)) + \xi_0 : y \in G,\ |x - y| \leqq h\}$, then $k(x) > 0$ for all x in G and $\int_G [k(x)]^{-1}\,dx < \infty$.*

Then for any ζ in the resolvent set of $(A_2 + V(x)I)$, the operator $(A_2 + V(x)I - \zeta I)^{-2}$ is an operator of Hilbert-Schmidt type while $(A_2 + V(x)I - \zeta I)^{-4}$ is an operator of finite trace class.

Proof of Theorem 27. The constants ξ_0 and h of the hypothesis of Theorem 27 are fixed. Choose a testing function $\lambda(x)$ infinitely differentiable in E^n with compact support in the open disk $|x| < h$. Suppose u lies in $D(A_2 + V(x)I) \cap R(G)$. Applying the estimation of the proof of Theorem 24 to the function u instead of w with the testing functions λ_ε replaced by the

single testing function $\lambda(x - x_0)$ we obtain, if we set $(A_2 + V(x)I)u = v$,

$$\text{(5.86)} \qquad \mathrm{Re}\,(\lambda^{2m}v, u) = \mathrm{Re}\,\big((A_2 + V(x)I)u, \lambda^{2m}u\big) \geqq c_0 \sum_{|\alpha| \leqq m} \|\lambda^m D^\alpha u\|^2_{0,2} - \\ -K \sum_{|\alpha|, |\beta| \leqq m, |\alpha| + |\beta| \leqq 2m-1} \|\lambda^{|\alpha|} D^\alpha u\|_{0,2} \cdot \|\lambda^{|\beta|} D^\beta u\|_{0,2}$$

where the norms in (5.86) are taken over the support of λ which is contained in the open set $G_h = G \cap \{x : |x| < h\}$. Estimating the last sum in (5.86) as in the proof of Theorem 24, we obtain

$$\text{(5.87)} \qquad \sum_{|\alpha| \leqq m} \|\lambda^m D^\alpha u\|^2_{L^2(G_h)} \leqq K\{\|v\|^2_{L^2(G_h)} + \|u\|^2_{L^2(G_h)}\}\,.$$

We assume that the constant ξ_0 has already been absorbed in the operator A and in the function V (without any loss of generality) so that $(A_2 + V(x)I)^{-1}$ exists on all of $L^2(G)$ and is a bounded operator on $L^2(G)$, while $\mathrm{Re}\,(A_2 u, u) \geqq 0$ for all u in $D(A_2)$. It follows from this last assumption that for u in $D(A_2 + V(x)I) \cap R(G)$,

$$\text{(5.88)} \qquad \mathrm{Re}\,\big((A_2 + V(x)I)u, u\big) \geqq \mathrm{Re}\,(V(x)u, u) \geqq k(x_0)\,\|u\|^2_{L^2(G_h(x_0))}\,.$$

If we take the limit on both sides of (5.88), since the first and last term are continuous on the graph of $D(A_2 + V(x)I)$, it follows that (5.88) is valid for all u in $D(A_2 + V(x)I)$. But (5.88) implies that for $v = (A_2 + V(x)I)u$, $u = (A_2 + V(x)I)^{-1}v$, we have

$$\text{(5.89)} \qquad \|u\|^2_{L^2(G_h)} \leqq K\,[k(x_0)]^{-1}\|v\|^2_{L^2(G)}\,.$$

It follows from (5.87) and Lemma 5 of Section 1, since $2m > n$, that if we choose λ to be identically one on the $\left(\frac{h}{2}\right)$-disk around x_0, we will have

$$\text{(5.90)} \qquad |u(x_0)|^2 \leqq K\{\|v\|^2_{L^2(G_h(x_0))} + [k(x_0)]^{-1}\|v\|_{L^2(G)}\}\,.$$

However, if v also lies in $D(A_2 + V(x)I)$, i.e. if u lies in $D\big((A_2 + V(x)I)^2\big)$, and if we set $w = (A_2 + V(x)I)v$, then (5.89) is valid for the pair v and w instead of u and v, and (5.90) becomes

$$\text{(5.91)} \qquad |u(x_0)|^2 \leqq K_1\,[k(x_0)]^{-1}\|(A_2 + V(x)I)^2 u\|^2_{L^2(G)}\,.$$

It follows from the Dunford-Pettis theorem applied to bounded subdomains of G that the transformation $(A_2 + V(x)I)^{-2}$ is an integral transformation of the form

$$\text{(5.92)} \qquad \{(A_2 + V(x)I)^{-2}u\}\,(x) = \int_G K(x, y)\,u(y)\,dy\,,$$

with

$$\text{(5.93)} \qquad \int_G |K(x, y)|^2\,dy \leqq K_1[k(x)]^{-1},$$

and K measurable on the product space $G \times G$. Integrating (5.93) with respect to x over G, we obtain

$$\text{(5.94)} \qquad \int_{G \times G} |K(x, y)|^2\,dx\,dy \leqq K_2\,,$$

$(A_2 + V(x)I)^{-2}$ is obviously a Hilbert-Schmidt operator, and by the fact that every product of two Hilbert-Schmidt operators is an operator of finite trace class, $(A_2 + V(x)I)^{-4}$ is an operator of finite trace class. The result for general ζ follows by standard arguments and Theorem 27 is proved.

Section 6: Resolvents of non-self-adjoint operators

If A is an essentially real elliptic differential operator on G, then so is its Hermitian part, $H = 1/2\,(A + A')$. Let S be the skew-Hermitian part of A, $S = (2i)^{-1}(A - A')$.

It is the purpose of the present section to study the essential spectrum of the operator A_p by studying the difference

$$S_\zeta = (A_p - \zeta I)^{-1} - (H_p - \zeta I)^{-1}, \tag{1}$$

for ζ in the intersection of the resolvent sets of A_p and H_p, under various assumptions on the behaviour of the coefficients of S at infinity in G. In particular, if the coefficients of S approach zero as $|x| \to \infty$ in G, we show that S_ζ is a compact operator in $L^p(G)$, and as a consequence, A_p and H_p have the same essential spectrum. Under stronger assumptions, S_ζ has an iterate of finite trace class. Further results as well as the detailed hypotheses are given in the discussion which follows.

Theorem 28. *Let A be an uniformly strongly elliptic differential operator of order $2m$ on an open subset G of E^n, with G uniformly regular of class C^{2m} and locally regular of class C^{4m}. Suppose that A is essentially real (i.e. that $a_\alpha(x)$ is a real-valued function on G for all α with $|\alpha| = 2m$) and that A satisfies the regularity condition* (IV) *of Section* 3. *If H and S are the Hermitian and skew-Hermitian parts of A, respectively, (i.e. $H = 1/2(A + A')$, $S = 1/2i(A - A')$), we suppose that the coefficients $s_\alpha(x)$ of $S = \sum_{|\alpha| < 2m} s_\alpha(x) D^\alpha$ approach zero as x goes to infinity in G. Then:*

(a) *The essential spectrum of A_p coincides with that of H_p for all p, $1 < p < \infty$. In particular if $p = 2$, the essential spectrum of A_2 is contained in a real half-line of the form $\{\xi : \xi \geqq \xi_0\}$.*

(b) *For ζ in the intersection of the resolvent sets of A_p and H_p, the operator $S_\zeta = (A_p - \zeta I)^{-1} - (H_p - \zeta I)^{-1}$ is a compact operator in $L^p(G)$.*

Before proceeding to the proof of Theorem 28, it is necessary to describe the precise sense to be given the term *essential spectrum* as used in part (a) of Theorem 28. For self-adjoint operators in Hilbert space, our usage coincides with the usual one in that context (e.g. [71]), but the latter usage has a number of alternative possible extensions to the case of general operators in a Banach space.

Definition 11. *Let T be a closed linear operator with dense domain in the Banach space X. The complex number ζ is said to lie in the essential spectrum of the operator T whenever at least one of the following conditions holds:*

(i) *The operator $(T - \zeta I)$ has a range which is not closed in X;*

(ii) $\bigcup_{r \geqq 0} N((T-\zeta I)^r)$ *is of infinite dimension, ($N(S)$ being the null-space of the operator S);*

(iii) *The point ζ is a limit point of the spectrum of T.*

Thus the complement in the spectrum of T of the essential spectrum of T consists precisely of all point spectrum ζ of finite multiplicity of T and of point spectrum of its adjoint T^*, both isolated in the spectrum of T.

Let $\sigma(T)$ denote the spectrum of T, $\mathrm{ess}(T)$ the essential spectrum of T. We shall consider in more detail the structure of T with respect to a complex number ζ_0 in $\sigma(T) - \mathrm{ess}(T)$. Let C be a closed rectifiable Jordan curve which contains ζ_0 in its interior but no other points of $\sigma(T)$. The bounded linear operator P_0 on X defined by

$$2\pi i\, P_0 = \int_C (\zeta I - T)^{-1}\, d\zeta\,, \tag{2}$$

is an idempotent linear mapping of X into itself such that $P_0(X) \subset D(T)$, $X_0 = P_0(X)$ is invariant under T and closed in X, and the spectrum of T_0, the restriction of T to X_0, consists of the single point ζ_0. Moreover, the range $R(T_0 - \zeta_0 I)$ of the operator $(T_0 - \zeta_0 I)$ in X_0 equals $R(T-\zeta_0 I) \cap X_0$, since if u lies in $D(T)$ while $(T-\zeta_0 I)u = v \in X_0$, then $(T_0 - \zeta_0 I)\, P_0 u = P_0(T-\zeta_0 I)u = v$, implies that v lies in $R(T_0 - \zeta_0 I)$. In particular, since ζ_0 does not lie in $\mathrm{ess}(T)$, $R(T-\zeta_0 I)$ is closed and therefore so is $R(T_0 - \zeta_0 I)$.

Let $N_0 = \bigcup_{r \geqq 0} N((T-\zeta I)^r)$, be the eigenspace of the transformation T with respect to ζ_0. By the assumption that $\zeta_0 \notin \mathrm{ess}(T)$, N_0 is of finite dimension and hence closed in X, and in particular $N_0 = N((T-\zeta_0 I)^r)$ for some fixed sufficiently large positive integer r. We assert first that $N_0 \subset X_0$. Indeed, suppose that we have shown that $N((T-\zeta_0 I)^{k-1}) \subset X_0$ for some $k \geqq 1$. Then for $u \in N((T-\zeta_0 I)^k)$, $u_0 = (T-\zeta_0 I)u$ lies in X_0. Let $w_\zeta = (\zeta I - T)^{-1} u$, for $\zeta \in C$. Then,

$$u_0 = Tu - \zeta_0 u = (Tu - \zeta I)u + (\zeta - \zeta_0)u\,,$$

which implies that

$$u = (\zeta - \zeta_0)^{-1} u_0 + (\zeta - \zeta_0)^{-1}(\zeta I - T)u\,,$$

and

$$w_\zeta = (\zeta I - T)^{-1} u = (\zeta - \zeta_0)^{-1}(\zeta I - T)^{-1} u_0 + (\zeta - \zeta_0)^{-1} u\,. \tag{3}$$

But for u_0 in X_0, $(\zeta I - T)^{-1} u_0 = v_0$ also lies in X_0, since the equation $(\zeta I - T)v_0 = u_0$ immediately implies that $(\zeta I - T)(P_0 v_0) = u_0$, and by the uniqueness of the inverse, $v_0 = P_0 v_0$. Integrating (3) with respect to ζ around the closed curve C, we obtain

$$P_0 u = u + \int_C (\zeta - \zeta_0)^{-1}(\zeta I - T)^{-1} u_0\, d\zeta\,, \tag{4}$$

and, since the second term on the right in (4) lies in X_0, u lies in X_0. Proceeding by recursion, it follows that $N_0 \subset X_0$.

We now assert that $N_0 = X_0$, under our present hypotheses. To establish this fact, we remark first that the transformation $T' = T_0 - \zeta_0 I$ of X_0 has

only zero in its spectrum, i.e. T' is quasi-nilpotent. In addition, $T'(N_0) \subset N_0$, while if $T'u \in N_0$, then u must lie in $N((T-\zeta_0 I)^{r+1}) = N_0$. Let X_1 be the quotient space X/N_0 with the usual Banach space structure, and let T_1 be the linear transformation of X_1 induced by T'_0, i.e.

$$T_1(u+N_0) = T'_0 u + N_0 \,. \tag{5}$$

Since for every $k \geqq 1$, T_1^k is the transformation of X_1 induced by $(T'_0)^k$ and since the norm of the induced transformation is not more than the norm of the inducing transformation, it follows that $\|T_1^k\| \leqq \|(T'_0)^k\|$, so that the spectral radius of the transformation T_1 must be zero since T'_0 is quasi-nilpotent. On the other hand, $R(T_1)$, the range of T_1 in X_1, is a closed subspace of X_1, since its inverse image in X_0 under the quotient mapping is $R(T'_0)+N_0$, $R(T'_0)$ is a closed subspace of X_0, and the sum of a closed subspace and a finite-dimensional subspace of a Banach space is always closed. Finally, T_1 has a trivial null-space, since $T_1(u+N_0) = N_0$ implies that $T'_0 u$ lies in N_0, and by our preceding remark, it follows that u must lie in N_0. Hence T_1 is a one-to-one mapping of X_1 onto the closed subspace $R(T_1)$ of X_1. Suppose X_1 is not trivial. By the open mapping theorem, it follows that there exists a constant $c>0$ such that $\|T_1 u\| \geqq c\|u\|$, for all u in X_1. But then by recursion, it follows that $\|T_1^k u\| \geqq c_k \|u\|$ for all u in X_1, i.e. $\|T_1^k\| \geqq c^k$ for all k, and hence $\|(T_1^k)\|^{1/k} \geqq c$ for all k. This contradicts the fact that T_1 is quasi-nilpotent, thus proving that X_1 must be trivial, and hence $X_0 = N_0$.

Let $P_1 = (I - P_0)$ be the complementary projection to P_0. P_1 maps $D(T)$ into itself, and if $X' = P_1(X)$, T' the restriction of T to $X' \cap D(T)$, then T' is a closed densely-defined linear mapping of $D(T') \subset X'$ into X'.

We remark now that $(T' - \zeta_0 I)^{-1}$ is a bounded linear mapping defined on all of X'. We give a simple direct proof of this fact although undoubtedly it may be derived from the spectral mapping theorem for closed operators. (For the latter, as well as a more detailed discussion of the operational calculus for closed operators, we refer to HILLE-PHILLIPS [34], pp. 199—211.) By the same argument as we applied in X_0, if v lies in $R((T-\zeta_0 I)) \cap X'$, then v lies in $R((T'-\zeta_0 I))$. The transformation $(T'-\zeta_0 I)$ of X' has a trivial null-space, since the null-space of $(T-\zeta_0 I)$ is contained in X_0. To establish our conclusion, it suffices therefore to show that the range of $(T-\zeta_0 I)$ contains all of X'. Let u be an element of X', and for ζ in C, let $w_\zeta = (\zeta I - T)^{-1} u$. Consider the element w of X' given by the formula

$$w = (2\pi i)^{-1} \int_C (\zeta_0 - \zeta)^{-1} (\zeta I - T)^{-1} u \, d\zeta \,.$$

Since $(T-\zeta_0 I)(\zeta I - T)^{-1} = -I + (\zeta - \zeta_0)(\zeta I - T)^{-1}$, it follows that w lies in $D(T)$ and that

$$(T-\zeta_0 I) w = (2\pi i)^{-1} \int_C (\zeta - \zeta_0)^{-1} u \, d\zeta + \int_C (\zeta I - T)^{-1} u \, d\zeta \,.$$

The second integral on the right equals zero, however, since u lies in X', so that $(T-\zeta_0 I) w = u$. Since u is an arbitrary element of X', it follows that $(T'-\zeta_0 I)^{-1}$ exists on all of X'.

Since X' is the null-space of the finite-dimensional projection P_0, we obtain immediately: the range of $(T-\zeta_0 I)$, for ζ_0 not in the essential spectrum of T, must be of finite co-dimension in X.

Let us now consider the Laurent expansion of the analytic operator-valued function $(\zeta I - T)^{-1}$ at its isolated singularity at ζ_0. We have

$$(\zeta I - T)^{-1} = \sum_{j=-\infty}^{+\infty} A_j (\zeta - \zeta_0)^{-j},$$

where the operator A_j is given by

$$A_j = (2\pi i)^{-1} \int_C (\zeta - \zeta_0)^{-j-1} (\zeta I - T)^{-1} \, d\zeta .$$

In particular, for $j = -r-1$,

$$A_{-r-1} = (2\pi i)^{-1} \int_C (\zeta - \zeta_0)^r (\zeta I - T)^{-1} \, d\zeta ,$$

while for u in $D(T^r)$,

$$(\zeta - \zeta_0)^r (\zeta I - T)^{-1} u = (T - \zeta_0 I)^r (\zeta I - T)^{-1} u + g(\zeta) u ,$$

where $g(\zeta)$ is analytic within C. Hence, for u in $D(T^r)$,

$$A_{-r-1} = (T - \zeta_0 I)^r \int_C (\zeta I - T)^{-1} u \, d\zeta = (T - \zeta_0 I)^r A_{-1} u ,$$

while $A_{-1} = P_0$ as defined above. Hence $A_{-r-1} u = (T - \zeta_0 I)^r P_0 u = 0$ for all u in $D(T^r)$. Since $D(T^r) = D((T - \zeta I)^r)$ for any ζ, the set $D(T^r)$ is dense in X, (for indeed $(T^* - \xi I)^r$ for any given ζ on C has a bounded inverse). Hence the bounded operator A_{-r-1} is zero on all of X. Since for $j < -r-1$, $A_j = (T - \zeta_0 I)^{r+1+j} A_{-r-1}$, all the A_j for $j < r$ must vanish. Thus for ζ_0 outside the essential spectrum of T, the function $(\zeta I - T)^{-1}$ has at most a pole at its isolated singularity ζ_0.

The results of our preceding remarks are summarized and completed in the following Lemma:

Lemma 17. *Let T be a closed operator densely defined in the Banach space X with finite-dimensional eigenspace for the complex number ζ_0. Then the point ζ_0 of the spectrum of T does not lie in the essential spectrum of T if and only if the resolvent $(T - \zeta I)^{-1}$ is analytic in the neighborhood of ζ_0 and has a pole at ζ_0. Under this hypotheses, $R(T - \zeta_0 I)$ has finite co-dimension in X.*

Proof of Lemma 17. We have already shown that if ζ_0 does not lie in the essential spectrum of T, then $(T - \zeta I)^{-1}$ has at most a pole at ζ_0.

Suppose conversely that $(T - \zeta I)^{-1}$ has a pole of order r at ζ_0. Then for a closed rectifiable Jordan curve C containing ζ_0 and no other points of the spectrum of T in its interior, we have

$$(2\pi I)^{-1} \int_C (\zeta - \zeta_0)^r (\zeta I - T)^{-1} \, d\zeta = 0 .$$

But as before, for u in $D(T^r)$,

$$\int_C (\zeta - \zeta_0)^r (\zeta I - T)^{-1} u \, d\zeta = (T - \zeta_0 I)^r P_0 u ,$$

where P_0 is defined by

$$P_0 u = \int_C (\zeta I - T)^{-1} u \, d\zeta \, .$$

But P_0 maps all of X into $D(T^r)$, so that $(T - \zeta_0 I)^r P_0 = 0$ by the density of $D(T^r)$ in X. In particular, P_0 is a idempotent mapping of X whose range, being contained in the eigenspace of T with respect to ζ_0, must be finite-dimensional since that eigenspace is finite-dimensional. The range of the complementary idempotent $(I - P_0)$ will be denoted as before by X'. By the same argument as above, X' is invariant under the mapping T' obtained by restricting T to $D(T) \cap X'$, and $(T' - \zeta_0 I)^{-1}$ is well-defined on all of X'. In particular, X' is contained in the range of $(T - \zeta_0 I)$, and the latter is the sum of X' and the image of $P_0(X)$ under T, which is finite dimensional. Since any finite-dimensional extension of a closed subspace of X is also closed, it follows that the range of $(T - \zeta_0 I)$ is a closed subspace of X. Since X' is obviously of finite co-dimension in X, $R((T - \zeta_0 I))$ has finite co-dimension in X. Thus all the conclusions of Lemma 17 are established.

The basic fact about perturbation of essential spectrum which we shall apply in our later discussion is contained in the following result:

Lemma 18. *Let T be a closed densely-defined operator in the Banach space X, C an operator in X such that $D(C) \supset D(T)$. Suppose that for some point ζ in the resolvent set of T, the operator $C(T - \zeta I)^{-1}$ is compact, $(T + C)$ has at least one point in its resolvent set, and the resolvent set of T is connected and dense in the complex plane.*

Then the closed operators T and $(T + C)$ have the same essential spectrum.

Proof of Lemma 18. We shall derive the conclusion of our Lemma from the theorem of SMULYAN [64] which asserts that if $C(\zeta)$ is an analytic operator-valued function on a connected open set F of the complex ζ plane whose values are compact operators on the Banach space X, then for any $\lambda_0 \neq 0$, one of the two following possibilities must hold:

(i) For every ζ in F, the operator $C(\zeta) - \lambda_0 I$ has a non-trivial null-space; or,

(ii) Except for a discrete set of values ζ_k in F, the operator $(C(\zeta) - \lambda_0 I)$ has a bounded inverse on all of X, while $(C(\zeta) - \lambda_0 I)^{-1}$ has a pole at each of the points ζ_k.

We consider first a point ζ_0 in the resolvent set of T. For any ζ in the resolvent set of T,

$$(T + C - \zeta I) = (T - \zeta I) + C = (I + C(T - \zeta I)^{-1})(T - \zeta I) \, .$$

Hence $(T + C - \zeta I)$ will have a bounded inverse defined on all of X if and only if $(I + C(\zeta))$ has one, where $C(\zeta)$ is the analytic operator-valued function $C(T - \zeta I)^{-1}$ defined on the resolvent set of T and having as values compact operators on X. (We remark that we have only assumed that $C(T - \zeta I)^{-1}$ is compact for one value of ζ in the resolvent set of T. If ζ_1 is such a value of ζ,

then for other points ζ in the resolvent set of T, it follows from the first resolvent equation,

$$(T-\zeta I)^{-1} = (T-\zeta_1 I)^{-1} + (\zeta-\zeta_1)(T-\zeta_1 I)^{-1}(T-\zeta I)^{-1},$$

that

$$C(T-\zeta I)^{-1} = C(T-\zeta_1 I)^{-1}[I + (T-\zeta I)^{-1}],$$

and hence $C(T-\zeta I)^{-1}$ is also a compact operator on X.)

Hence, applying the result of SMULYAN stated above with $\lambda_0 = -1$, it follows that either $(T+C)$ has only isolated spectrum on the resolvent set of T with its resolvent $(T+C-\zeta I)^{-1}$ having a pole and a finite dimensional eigenspace at each point of its spectrum in this set, or the resolvent set of T is contained in the point spectrum of $(T+C)$. In the first case, no point of the resolvent set of T falls within the essential spectrum of $(T+C)$. In the second case, since by assumption the resolvent set of T is dense in the plane, we should have the spectrum of $(T+C)$ dense in the plane. Since the spectrum of $(T+C)$ is closed, it would follow that the resolvent set of $(T+C)$ is empty, contradicting one of the assumptions of the Lemma. Thus, the second case being excluded, the first case must hold, and the resolvent set of T is contained in the complement of the essential spectrum of $(T+C)$.

A modification of the above argument may be applied to the case of a point ζ_0 in the spectrum of T but not in the essential spectrum of T. To carry through this argument, we apply a simple device, for which we are indebted to a conversation with J. T. SCHWARTZ. Let P_0 be the projection of X on the eigenspace of $(T-\zeta_0 I)$ as constructed in the discussion leading up to the proof of Lemma 17. Then the operator $(T+C)$ may be written as $(T+P_0)+(C-P_0)$. Since ζ_0 does not lie in the essential spectrum of T, P_0 has a finite-dimensional range and the operator $(C-P_0)$ is compact. Thus, instead of regarding $(T+C)$ as the perturbation of T by C, we may regard it as the perturbation of $(T+P_0)$ by $(C-P_0)$. But the operator $(T+P_0-\zeta_0 I)$ has a bounded inverse defined on all of X, since the subspaces $X_0 = P_0(X)$ and $X' = (I-P_0)(X)$ are invariant under this operator, on X' the operator $(T+P_0-\zeta_0 I)$ coincides with $(T'-\zeta_0 I)$ which has a bounded inverse on all of X', and on the finite dimensional space X_0 the eigenvalues of $(T+P_0-\zeta_0 I)$ are all equal to 1. Thus we may apply the argument of the preceding paragraph and assert that the spectrum of $(T+C)$ is a discrete set in the neighborhood of ζ_0 unless the whole neighborhood is composed of point spectrum of $(T+C)$. But since the resolvent set of $(T+C)$ is non-empty, it follows as before that this second contigency is impossible so that ζ_0 is an isolated point of the spectrum of $(T+C)$ if it lies at all in that spectrum. The conclusion that ζ_0 does not lie in the essential spectrum of $(T+C)$ also follows from the above dichotomy in the result of SMULYAN.

Thus we see from the above arguments that the essential spectrum of T contains the essential spectrum of $T+C$. Moreover, the resolvent set of $T+C$ is obtained from the resolvent set of T by removing at most a countable set of points whose only limit points are in the essential spectrum of T. It follows

immediately that the resolvent set of $T + C$ is connected and everywhere dense in the plane, since these properties hold for the resolvent set of T. But then all our hypotheses on T and C are valid for the operators $T + C$ and $(-C)$. Since T may be written as $(T + C) - C$, it follows from the above argument that the essential spectrum of $T + C$ contains the essential spectrum of T. It follows that the two essential spectra are the same, and Lemma 18 is proved.

A modification of the above proof yields the following conclusion:

Lemma 19. Let T be a closed densely-defined operator in the Banach space X, C an operator in X such that $D(C) \supset D(T)$. Suppose that for some point ζ in the resolvent set of T, the operator $C(T - \zeta I)^{-1}$ is compact. Suppose that in addition the point spectra of T and $T + C$ are nowhere dense. Then the essential spectra of T and $T + C$ are identical.

Proof of Lemma 19. The only use made in the proof of Lemma 18 of the hypothesis that the resolvent set of T is connected and dense while the resolvent set of $T + C$ is non-empty was to avoid the possibility of dense point spectra for T or for $T + C$ in the neighborhood of a point in the complement of one of the essential spectra. Since under the hypotheses of Lemma 19, the point spectra of both operators are nowhere dense, we do not need the other hypotheses to ensure the non-density of point spectra.

Lemma 20. Let T be a closed densely-defined linear operator in the Banach space X, T^* its adjoint operator in the adjoint space X^*. Then the essential spectrum of T and T^* are identical.

Proof of Lemma 20. By Lemma 17, ζ_0 lies in the complement of the essential spectrum of T if and only if ζ_0 is an isolated point of the spectrum (or in the resolvent set), the eigenspace of T with respect to ζ_0 is finite-dimensional, and $(T - \zeta I)^{-1}$ has at worst a pole at ζ_0.

Similarly, ξ_0 will lie in the complement of the essential spectrum of T^* if and only if ξ_0 is an isolated point of the spectrum of T^* (or lies in the resolvent set of T^*), the eigenspace of T^* with respect to ξ_0 is of finite dimension, and $(T^* - \xi I)^{-1}$ has at worst a pole at ξ_0. (We remark that the proof of Lemma 17 for T^* may be carried through as before, with the additional remark that though the domain of $(T^* - \xi_0 I)^r$ is only weak*-dense in X^*, the corresponding operator A_{-r-1} as defined above will be weak*-continuous in the present circumstances.)

However, ζ lies in the resolvent set of T if and only if ξ lies in the resolvent set of T^*, and $((\zeta I - T)^{-1})^* = (\xi I - T^*)^{-1}$ for such ζ. Hence $(\zeta I - T)^{-1}$ has a pole at ζ_0 if $(\xi I - T^*)^{-1}$ has a pole at ξ_0. Hence we need to consider only the equivalence of the conditions that T should have a finite-dimensional eigenspace with respect to ζ_0 and that T^* should have a finite-dimensional eigenspace with respect to ξ_0. But the null-space of $(T^* - \xi_0 I)^r$, which is the eigenspace of T^* with respect to ξ_0 if $(\zeta I - T^*)^{-1}$ has a pole of order r at ξ_0, is precisely the annihilator of the range of $(T - \zeta_0 I)^r$, which by the latter part of the proof of Lemma 17 includes all of the subspace X' of finite co-dimension in X. Hence the eigenspace of T^* at ξ_0 is of finite dimension if ζ_0 does not lie in the essential

spectrum of T. The converse follows by the same argument, reversing the roles of T and T^*.

Proof of Theorem 28. Using Lemmas 17 through 20, we first derive the assertion (a) of Theorem 28 from a variant of (b), and establish the latter in the second stage of the argument.

Proof of (a). Since $A' = H - iS$, A' has the same symmetric part H as A and the negative anti-symmetric part $(-S)$. Hence the hypotheses of Theorem 28 are fulfilled for A' if they are satisfied for A itself. In addition, if $q = p(p-1)^{-1}$ $(p, q > 1)$, it follows from Theorem 16 of Section 4 that $(A_p)^* = A'_q$. By Lemma 20, the essential spectrum of A_p therefore coincides with that of A'_q while the essential spectrum of H_p coincides with that of H_q. It thus suffices to consider only the case $p \leqq 2$.

We now observe that the hypotheses of Lemma 19 are satisfied for the operators $T = H_p$ and $T + C = A_p$.

The hypothesis that $C(T - \zeta I)^{-1}$ should be compact is proved below. By Theorems 10 and 15(c), the point spectra of H_p and A_p respectively are contained in the point spectra of H_2 and A_2, respectively. Since H_2 is self-adjoint in $L^2(G)$ by Theorem 16, its spectrum is contained in the real axis and hence nowhere dense in the complex plane. Moreover, by Theorem 15(c), the spectra of H_2 and A_2 are contained in a half-plane. In particular, the spectrum of H_2 does not disconnect the plane, and the resolvent set of A_2 is non-empty. Applying Lemma 18 together with the assumption (b) to the operators $T = H_2$ and $T + C = A_2$, it follows from that Lemma that A_2 and H_2 have the same essential spectrum. In particular, the spectrum of A_2 off the real line is isolated. Thus the spectrum of A_2, and more particularly the point spectrum of A_2 does not disconnect the plane. Since the point spectrum of A_p is contained in that of A_2, the point spectrum of A_p does not disconnect the plane. Therefore, all the hypotheses of Lemma 19 are satisfied, and as a result A_p and H_p have the same essential spectrum for $p \leqq 2$. By our previous remarks, this suffices to establish (a), and we now turn to the proof of assertion (b) of Theorem 28.

Proof of (b). We begin with the proof of the fact that for ζ in the resolvent set of H_p, the operator $S(H_p - \zeta I)^{-1}$ is a compact linear operator in $L^p(G)$. (A similar proof shows that for ζ in the resolvent set of A_p, the operator $S(A_p - \zeta I)^{-1}$ is compact. If the union of these two resolvent sets is empty, both assertions (a) and (b) are vacuously true. In any other case, it is the compactness of one of the two operators $S(A_p - \zeta I)^{-1}$ and $S(H_p - \zeta I)^{-1}$ which we have applied in the proof of (a).)

Let N be an arbitrary positive integer. We define an operator C_N by the prescription,

$$C_N u = M_N S(H_p - \zeta I)^{-1} u ,$$

where M_N is the multiplication operator in $L^p(G)$ of multiplying by the characteristic function of the set $B_N = \{x : |x| \leqq N\} \cap G$.

Let $v = (H_p - \zeta I)^{-1} u$. Then $(H_p - \zeta I) v = u$. Then the differential operator $H_p - \zeta I$ satisfies the conditions of Theorem 2, and for $\|u\| \leqq 1$, we have

$\|v\| \leqq M$, and therefore

$$\|v\|_{W^{2m,p}(G)} \leqq M_1 .$$

For $|\alpha| < 2m$, $D^\alpha(H_p - \zeta I)^{-1}$ is a bounded mapping of $L^p(G)$ into $W^{1,p}(G)$, and its restriction to B_N is a compact mapping into $L^p(B_0)$ by Lemma 5(b). The multiplication of the last-obtained element by the bounded coefficient $s_\alpha(x)$ of S still yields a compact operator from $L^p(G)$ into $L^p(B_N)$, and if we combine this last operator with the natural injection of $L^p(B_N)$ into $L^p(G)$, we see finally that C_N is a compact operator for each N. Consider, on the other hand, $(S(H_p - \zeta I)^{-1} - C_N)u = C'_N u$. The operator $C'_N = M'_N S(H_p - \zeta I)^{-1}$, where M'_N is the multiplication operator by the characteristic function of the complement of B_0. We have,

$$\|C'_N u\| \leqq K\, s(N)\, \|u\| ,$$

where K is a constant independent of N, while $s(N)$ is the maximum of the suprema of $|s_\alpha(x)|$ taken over $G - B_N$. By hypothesis, $s(N) \to 0$ as $N \to \infty$. Hence $\|S(H_p - \zeta I)^{-1} - C_N\| \to 0$ as $N \to \infty$. It follows, since the uniform limit of compact operators is compact, that $S(H_p - \zeta I)^{-1}$ is compact.

Finally, consider $S_\zeta = (A_p - \zeta I)^{-1} - (H_p - \zeta I)^{-1}$ for ζ in the intersection of the resolvent sets of A_p and H_p. The range of S_ζ is contained in the common domains of $(A_p - \zeta I)$ and $(H_p - \zeta I)$. We note that

$$(A_p - \zeta I) S_\zeta = I - (H_p + S - \zeta I)(H_p - \zeta I)^{-1} = S(H_p - \zeta I)^{-1},$$

i.e. $S_\zeta = (A_p - \zeta I)^{-1} S(H_p - \zeta I)^{-1}$. Since $S(H_p - \zeta I)^{-1}$ is compact, it follows that S_ζ is compact. Thus the proof of Theorem 28 is complete.

Our next task is a closer study of the operator S_ζ under stronger hypotheses on the coefficients of the skew-Hermitian part of the elliptic operator A at infinity. We preface this study with a brief summary of the definition of operators of finite trace class in a general Banach space X (following RUSTON [55] and GROTHENDIECK [32]), together with some sufficient conditions for an operator to be of finite trace class if the space X is $L^p(G)$ and more particularly if $X = L^2(G)$.

Let T be an operator of finite rank in the Banach space X, i.e. with a finite dimensional range R_0. If we choose a base $\{x_1, \ldots, x_r\}$ for R_0, then for every x in X, $Tx = \sum_{j=1}^{r} \lambda_j(x) x_j$, where the $\lambda_j(x)$ are uniquely defined for each x, and, since the mapping $x \to \lambda_j(x)$ is obviously a closed linear mapping from X into E^1, there exists a unique element x_j^* in X^* such that $\lambda_j(x) = \langle x, x_j^* \rangle$ for all x in X. There thus corresponds to T the element $\sum_{j=1}^{r} x_j \otimes x_j^*$ in $X \times X^*$. The trace norm of T, $\|\|T\|\|$, is defined to be $\inf \left\{ \sum_j \|x_j\|_X \|x_j^*\|_{X^*} \right\}$, where the infimum is taken over all the various possible representations of T in terms of the form $\sum_j x_j \otimes x_j^*$. It is easy to verify that $\|T\| \leqq \|\|T\|\|$ for all operators T of finite rank.

Definition 12. *A linear operator T in the Banach space X is of finite trace class if there exists a sequence $\{T_k\}$ of operators with finite rank in X such that $\|T_k - T_j\| \to 0$ as $j, k \to \infty$ while $T_k x \to Tx$ for each x in X.*

If we are given a sequence of operators $\{T_j\}$ of finite rank which form a Cauchy sequence in trace norm, i.e. $\|T_j - T_k\| \to \infty$ as $j, k \to \infty$, they also form a Cauchy sequence in the ordinary operator norm and hence converge in operator norm to an operator T on X which is obviously of finite trace class. It follows immediately that the linear space of operators of finite trace class, on which the trace norm $\|T\|$ may be defined as $\lim_j \|T_j\|$ for the approximating sequence $\{T_j\}$ of Definition 12, forms a Banach space with respect to this trace norm. We denote this space by $\mathscr{L}_t(X)$, and note that for $T \in \mathscr{L}_t(X)$, $\|T\| \leqq \|T\|$.

Properties of the operators in $\mathscr{L}_t(X)$ which are analogous to corresponding properties of the operators of finite trace class in a Hilbert space, have been studied by RUSTON [55], GROTHENDIECK [32], and others. Our principal concern in the present discussion of $\mathscr{L}_t(X)$ is to obtain sufficient conditions for an operator T to lie in $\mathscr{L}_t(X)$ and to obtain bounds for $\|T\|$ in such cases.

Lemma 21. (a) *Let $X = L^2(G)$, with G an open set of E^n, and let $T = S^2$, where S is an operator in $L^2(G)$ for which there exists a function f from $L^2(G)$ such that for almost all x in G,*

$$|Su(x)| \leqq f(x)\,\|u\|_{L^2(G)} \tag{6.6}$$

for all u in $L^2(G)$. Then T is an operator of finite trace class in $L^2(G)$ and S is a Hilbert-Schmidt operator.

(b) *Let $X = L^p(G)$, $1 < p < \infty$. Then T is of trace class if T is the integral operator in $L^p(G)$ defined by*

$$Tu(x) = \int_G K(x, y)\,u(y)\,dy\,, \tag{6.7}$$

where for the kernel $K(x, y)$, we have

$$K(x, y) = \sum_j c_j(x)\,K_j(x, y)\,d_j(y)\,, \tag{6.8}$$

with

$$K_j(x, y) = \sum_{\lambda, \xi} k_{j,\lambda,\xi,1} i(\langle \lambda, x\rangle + \langle \xi, y\rangle), \left(\langle \lambda, x\rangle = \sum_{k=1}^{n} \lambda_k x_k\right) \tag{6.9}$$

and

$$p_j = \sum_{\lambda, \xi} |k_{j,\lambda,\xi}| < \infty\,, \tag{6.10}$$

while

$$\sum_j p_j \|c_j\|_{L^p(G)} \cdot \|d_j\|_{L^q(G)} < \infty\,, \quad (q = p(p-1)^{-1})\,.$$

Remark. In the application of Lemma 21(b), the assumption (6.9) on the existence of an absolutely convergent Fourier series for K_j with the corresponding bound on the sum p_j of the absolute values of its Fourier coefficients may be replaced by the assumption that $K(x, y)$ is of sufficiently high differen-

tiability class on $\{\mathrm{Sup}(c_j) \times \mathrm{Sup}(d_j)\} \cap G \times G$, that the latter set is smooth for each j, and finally that one has uniform regularity and boundedness assumptions on the sets $\{\mathrm{Sup}(c_j) \times \mathrm{Sup}(d_j)\} \cap (G \times G)$. In the present discussion, to avoid further expansion of the length of the present paper, we shall restrict our attention to the case $p = 2$, leaving the related discussion for general p to another place.

Proof of Lemma 21(a). By (6.6), S is a bounded linear mapping from $L^2(G)$ into $L^\infty(G)$. By the Dunford-Pettis theorem, there exists a measurable kernel $K(x, y)$ on $G \times G$ with

$$n_x = \int_G |K(x, y)|^2 \, dy < \infty \,, \tag{6.11}$$

for almost all x in G, such that

$$Su(x) = \int_G K(x, y) \, u(y) \, dy \,. \tag{6.12}$$

It follows from (6.6), moreover, that for almost all x in G,

$$n_x \leqq [f(x)]^2 . \tag{6.13}$$

Since f lies in $L^2(G)$, it follows from the Fubini theorem that

$$\int_G \int_G |K(x, y)|^2 \, dx \, dy \leqq \int_G [f(x)]^2 \, dx < \infty \,. \tag{6.14}$$

In particular, S is an operator of Hilbert-Schmidt type, and it follows by a well-known argument that $T = S^2$ is an operator of finite trace class.

Proof of Lemma 21(b). We define

$$K^{(n)}(x, y) = \sum_{j \leqq n} c_j(x) \, K_j(x, y) \, d_j(y) \,.$$

By the definition of the trace norm, (for $m \geqq n$)

$$\| |K^{(n)}(x, y) - K^{(m)}(x, y)| \| \leqq \sum_{n < j \leqq m} \| |c_j(x) \, K_j(x, y) \, d_j(y)| \| \,,$$

while,

$$\| |c_j(x) \, K_j(x, y) \, d_j(y)| \| \leqq \sum_{\lambda, \xi} |k_{j, \lambda, \xi}| \, \| |c_j(x) e^{i\langle \lambda, x \rangle} \, d_j(y) e^{i\langle \xi, y \rangle}| \| \leqq$$

$$\leqq \|c_j\|_{L^p(G)} \, \|d_j\|_{L^q(G)} p_j \,.$$

Since the series (6.10) is convergent, it follows that the sequence $K^{(n)}(x, y)$ forms a Cauchy sequence in trace norm in $L^p(G)$ converging strongly to $K(x, y)$. Thus $K(x, y)$ is an operator of finite trace class.

We now return to the consideration of the operator $S_\zeta = (A_2 - \zeta I)^{-1} - (H_2 - \zeta I)^{-1}$. We have already shown in the present section that if the coefficients of the skew-Hermitian part of the elliptic operator A approach zero as $x \to \infty$ in G, then S_ζ is compact. We now turn to the proof that S_ζ has an iterate of finite trace class under sharper conditions on the behaviour of these coefficients at infinity.

Theorem 29. *Let* $A = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha$ *be an uniformly strongly elliptic differential operator of order* $2m$ *on an open subset* G *of* E^n *with* G *uniformly*

regular of class C^{2m} and locally regular of class C^{4m}. Suppose that A is essentially real and satisfies the regularity condition (IV) *of Section* 3 *on its coefficients. Let* $S = (2i)^{-1}(A - A') = \sum_{|\beta|<2m} s_\beta(x) D^\beta$ *be the skew-Hermitian part of A and consider the operator*

$$S_\zeta = (A_2 - \zeta I)^{-1} - (H_2 - \zeta I)^{-1}$$

where H is the Hermitian part of A, A_2 and H_2 are the realizations of A and H respectively in $L^2(G)$ under null Dirichlet boundary conditions.

Suppose that for each β with $|\beta| < 2m$, s_β lies in $L^2(G) \cap L^\infty(G)$. Let r be an integer such that

$$r\left(\frac{2m+1}{n}\right) > 1 + \frac{2m+2}{n}.$$

Then S_ζ^r is an operator of finite trace class in $L^2(G)$.

Proof of Theorem 29. As before, we assume in Theorem 29 that ζ lies in the intersection of the resolvent sets of A_2 and H_2. In particular, $(A_2 - \zeta I)^{-1}$ and $(H_2 - \zeta I)^{-1}$ are well-defined on all of $L^2(G)$. For v in $L^2(G)$, let $u = (A_2 - \zeta I)^{-1} v$. Then

$$v = (A_2 - \zeta I)u = (H_2 - \zeta I)u + i\,Su\,,$$

and, applying $(H_2 - \zeta I)^{-1}$ to the first and last expressions in the chain of equations,

$$(H_2 - \zeta I)^{-1} v$$
$$= u + i(H_2 - \zeta I)^{-1} Su = (A_2 - m\zeta I)^{-1} v + i(H_2 - \zeta I)^{-1}\, S(A_2 - \zeta I)^{-1} v\,,$$

i.e.

$$S_\zeta = (A_2 - \zeta I)^{-1} - (H_2 - \zeta I)^{-1} = -i(H_2 - \zeta I)^{-1}\, S(A_2 - \zeta I)^{-1}.$$

It thus suffices to prove that the r-th iterate of the operator $(H_2 - \zeta I)^{-1} S(A_2 - \zeta I)^{-1}$ is an operator of finite trace class in $L^2(G)$.

It follows from Theorem 2 and a boot-strap argument on p, as before, that if v lies in $L^2(G) \cap L^p(G)$, $p \geqq 2$, then $(A_2 - \zeta I)^{-1} v$ lies in $W^{2m,p}(G)$ and we have the inequality

$$\|(A_2 - \zeta I)^{-1} v\|_{W^{2m,p}(G)} \leq C_p\{\|v\|_{L^p(G)} + \|v\|_{L^2(G)}\}.$$

Similar conclusions hold for $(H_2 - \zeta I)^{-1}$.

The operator $(H_2 - \zeta I)^{-1} S(A_2 - \zeta I)^{-1}$ whose iterates we wish to consider may be written in the form

$$\sum_{|\beta|<2m} (H_2 - \zeta I)^{-1} s_\beta(x) \cdot D^\beta (A_2 - \zeta I)^{-1}. \tag{6.15}$$

Let us consider the term of (6.15) corresponding to some given index β. If v lies in $L^2(G) \cap L^p(G)$, $p \geqq 2$, it follows from the remarks of the last paragraph that $(A_2 - \zeta I)^{-1}$ lies in $W^{2m,p}(G)$ and hence that $D^\beta(A_2 - \zeta I)^{-1} v$ lies in $W^{2m-|\beta|,p}(G)$. By Lemma 5 of Section 1, we see that $D^\beta(A_2 - \zeta I)^{-1} v$ lies in $L^s(G)$ for $(p^{-1} - (2m - |\beta|)n^{-1}) \leq s^{-1} \leq 2^{-1}$. Since by assumption $s_\beta(x)$ lies in $L^\infty(G)$, it follows that $s_\beta D^\beta (A_2 - \zeta I)^{-1} v$ also lies in $L^s(G)$.

Since $(p^{-1}-(2m-|\beta|)n^{-1})$ is less than p^{-1}, it is less than 2^{-1}. Applying $(H_2-\zeta I)^{-1}$ and the conclusion of the preceding paragraph, we see that $(H_2-\zeta I)^{-1}\, s_\beta(x)\, D^\beta (A_2-\zeta I)^{-1} v$ lies in $W^{2m,s}(G)$ and hence in $L^f(G)$ with

$$(p^{-1}-(4m-|\beta|)n^{-1} \leqq f^{-1} \leqq 2^{-1}$$

with its norm in $L^f(G)$ bounded by a constant multiple of the sum of the L^p and L^2 norms of v. If f_0 is the minimum of the upper limits of the f-intervals obtained for the various values of β as above, we conclude that $(H_2-\zeta I)^{-1}$ $S(A_2-\zeta I)^{-1}$ maps $L^p(G)\cap L^2(G)$ into $L^{f_0}(G)$ with corresponding bounds.

Applying $(H_2-\zeta I)^{-1}\, S(A_2-\zeta I)^{-1}$, r_1 times, we see that its r_1-th iterate will be a bounded mapping of $L^2(G)$ into $L^\infty(G)$ provided that

$$(2^{-1}-r_1(4m-|\beta|)n^{-1})<0$$

for all β. If we apply $(H_2-\zeta I)^{-1}\, S(A_2-\zeta I)^{-1}$ once more, noting that by an argument similar to the above $(A_2-\zeta I)^{-1}$ maps $L^2(G)\cap L^\infty(G)$ into itself, it follows that the mapping $s_\beta(A_2-\zeta I)^{-1}\, S_\zeta^{r_1}$ satisfies the conditions on S of Lemma 21(a) and is therefore a Hilbert-Schmidt mapping. Hence so is $(H_2-\zeta I)^{-1}\, s_\beta(A_2-\zeta I)\, S_\zeta^{r_1} = S_\zeta^{r_1+1}$. Thus $S_\zeta^{2r_1+2} = S_\zeta^r$ is an operator of finite trace class.

Theorem 30. *Let A be an essentially real, uniformly elliptic differential operator of order $2m$ satisfying the regularity condition* (IV) *of Section* 3 *on the open subset G of E^n, where G is assumed to be uniformly regular of class C^{2m} and locally regular of class C^{4m}. Suppose for the skew-Hermitian part S of A, $S=\sum_{|\beta|<2m} s_\beta(x)D^\beta$, that $|D^\gamma s_\beta|\leqq M$ for $|\gamma|\leqq|\beta|$, and that each s_β lies in $L^2(G)$. Let A_2 be the realization of A in $L^2(G)$ under null Dirichlet boundary conditions.*

Then for $(r-3)>nm^{-1}$, $(A_2-\zeta I)^{-r}=H_1+iH_2$, where H_1 and H_2 are self-adjoint and H_2 is an operator of finite trace class.

Proof of Theorem 30. As we have shown during the proof of Theorem 29, $(A_2-\zeta I)^{-1}=(H_2-\zeta I)^{-1}+S_\zeta$, where $S_\zeta=-i(H_2-\zeta I)^{-1}\, S(A_2-\zeta I)^{-1}$. Thus, $(A_2-\zeta I)^{-r}$ is the sum of terms of the form

$$P=(H_2-\zeta I)^{-j_1}\, S_\zeta^{k_1}(H_2-\zeta I)^{-j_2}\, S_\zeta^{k_2}\ldots(H_2-\zeta I)^{-j_s}\, S_\zeta^{k_s},$$

where $j_t, k_t\geqq 0$, $\sum_t (j_t+k_t)=r$. Since the adjoint of an operator of finite trace class is also of finite trace class, as well as the sum of two such operators, and since $(H_2-\zeta I)^{-r}$ is self-adjoint, to show that H_2 is of finite trace class it suffices to show that each of the above products is an operator of finite trace class whenever at least one of the exponents k_t is different from zero.

Each P defined above is of the form

$$P=\left(\prod_{j=1}^{k(r)} T_j\right) S_\zeta B$$

or

$$P=B S_\zeta \left(\prod_{j=1}^{k(r)} T_j\right) \quad ; \qquad k(r)=\begin{cases}\dfrac{r}{2}\,, & r \text{ even}\\[2ex] \dfrac{r-1}{2}\,, & r \text{ odd}\,,\end{cases}$$

where T_j for each j is either $(H_2 - \zeta I)^{-1}$ or S_ζ, and B is a bounded operator in $L^2(G)$. Since the composition of an operator of finite trace class with a bounded operator in either order is still an operator with finite trace, it suffices to assume that $B = I$.

Expanding S_ζ on the terms of S, without loss of generality we may restrict our attention to terms of the form

i) $$Q = \left(\prod_{j=1}^{k(r)} T_j \right) (H_2 - \zeta I)^{-1} s_\beta(x) I$$

or

ii) $$Q = s_\beta(x) D^\beta (A_2 - \zeta I)^{-1} \left(\prod_{j=1}^{k(r)} T_j \right),$$

where each of the T_j is now either $(H_2 - \zeta I)^{-1}$ or $(H_2 - \zeta I)^{-1} s_\beta D^\beta (A_2 - \zeta I)^{-1}$, for some index β with $|\beta| < 2m$.

The finite trace property for Q of the form (ii) follows by a trivial variant of the proof of Theorem 29. The corresponding property for Q of the form (i) follows by consideration of Q^*, the adjoint of Q, which is easily seen to be of the form

$$Q^* = \bar{s}_\beta(x) (H_2 - \bar{\zeta} I)^{-1} \prod_{j=1}^{k(r)} T_j^* ,$$

where T_j^*, by the results on adjoints in Section 3, is either $(H_2 - \bar{\zeta} I)^{-1}$ or $(A_2 - \bar{\zeta} I)^{-1} D^\beta \{\bar{s}_\beta(x) (H_2 - \bar{\zeta} I)^{-1}\}$, which by the boundedness hypothesis on the coefficients $s_\beta(x)$ and their derivatives is again a sum of operators of the same general type as the original T_j. We may apply the argument of the proof of Theorem 29 once more to obtain the conclusion of the present theorem.

Remark. By a more detailed study of the local bounds upon the mappings $(A_2 - \zeta I)^{-1}$ it is possible to obtain results of the type of Theorem 30 without any assumptions upon the derivatives of the coefficients $s_\beta(x)$. (Such results are summarized in the writer's note [8].) We shall consider such results in another place to avoid any further expansion of the present discussion.

Section 7: Second-order operators

In the course of our argument in the preceding sections, we have already remarked that certain results which we have obtained could be significantly sharpened for second-order operators. We shall now investigate this case in detail.

Lemma 22. *Let A be a differential operator of order two,*

$$A = \sum_{j,k=1}^{n} a_{jk}(x) D_{jk} + \sum_j b_j(x) D_j + c(x) I, \; (a_{jk} = a_{kj};\; D_{jk} = D_j D_k) ,$$

and suppose that a_{jk} is uniformly continuous on G for all j and k, and that the coefficients of A and its formal adjoint A' lie in $L^\infty(G)$.

Then A may be written in the generalized divergence form

(7.1) $$A u = \sum_{j,k} D_j (a_{jk}(x) D_k u) + \sum_j b_j'(x) D_j u + c'(x) u ,$$

where the functions b_j' and c lie in $L^\infty(G)$.

Proof. Operating on functions in $C^\infty(G)$, A' may be written in general as a differential operator with distribution coefficients by the formula

$$A'u = \sum_{j,k} D_j D_k(\bar{a}_{jk}(x)u) + \sum_j D_j(\bar{b}_j(x)u) + \bar{c}(x)u\,,$$

which by an application of the Leibniz formula may be written as

$$A'u = \sum_{j,k} \bar{a}_{jk}(x) D_j D_k u + \sum_j \Big\{2\sum_k D_k(\bar{a}_{jk}) + \bar{b}_j\Big\} D_j u + \\ + \Big\{\sum_{j,k} (D_j D_k \bar{a}_{jk}) + \sum_j (D_j \bar{b}_j) + \bar{c}\Big\} u\,.$$

By hypothesis we may write A' in the form

$$A'u = \sum_{j,k} c_{jk}(x) D_j D_k u + \sum_j e_j(x) D_j u + f(x)u\,,$$

where c_{jk}, e_j, and f lie in $L^\infty(G)$. Comparing corresponding coefficients in the two expressions for A', we see that

$$c_{jk}(x) = \bar{a}_{jk}(x)\,,$$
$$e_j(x) = \bar{b}_j(x) + 2\sum_k D_k(\bar{a}_{jk})$$
$$f(x) = \sum_{j,k} (D_j D_k \bar{a}_{jk}) + \sum_j (D_j \bar{b}_j) + \bar{c}(x)\,.$$

On the other hand,

$$Au - \sum_{j,k} D_j(a_{jk} D_k u) = -\sum_k \sum_j (D_j a_{jk}) D_k u + \sum_j b_j D_j u + c(x)u\,,$$

and the coefficients of this last differential operator of first order, being linear combinations with constant coefficients of b_j, c, and the e_j, must all lie in $L^\infty(G)$.

Lemma 23. *Let A be an essentially real, uniformly elliptic differential operator of order two with uniformly continuous top order coefficients on an open set G regular of class C, such that A and its formal adjoint A' have essentially bounded coefficients.*

Then for every even integer $p = 2q$, there exists a constant K such that for all real-valued u in $D(A_q)$, and real ξ we have

$$\|(A_p + \xi I)u\|_{L^p(G)} \geq (\xi - K)\,\|u\|_{L^p(G)}\,.$$

Proof of Lemma 23. By Lemma 22, we may assume A written in the form (7.1) with its coefficients b_j' and c' essentially bounded in G. By Lemma 10, we may assume that u lies in $C^2(\bar{G})$ and has compact support. Consider

$$(Au,\, u^{2q-1}) = \int_G \Big\{\sum_{j,k} D_j(a_{jk}(x) D_k u) + \sum_j b_j''(x) D_j u + c''(x)u\Big\} u^{2q-1}\,dx$$
$$= \sum_{j,k} \int_G a_{jk}(x)\,(2q-1)\,u^{2q-2} D_j u\, D_k u\,dx + \\ + \sum_j \int_G b_j''(x) D_j u\, u^{2q-1}\,dx + \int_G c''(x) u^{2q}\,dx\,.$$

Because of the uniform ellipticity of A and the fact that the coefficients $a_{jk}(x)$ are real-valued, it follows that there exists a constant c_0 independent of

both u and p such that

$$\sum_{j,k} \int_G a_{jk}(x) u^{2q-2} D_j u \, D_k u \, dx \geqq c_0 \int_G u^{2q-2} \sum_j |D_j u|^2 \, dx .$$

By the essential boundedness of $c''(x)$, we have

$$\left| \int_G c''(x) u^{2q} \, dx \right| \leqq C \|u\|^{2q}_{L^{2q}(G)}$$

with a suitable constant C independent of p and u.

By the Schwarz inequality and the essential boundedness of the functions $b_j''(x)$, there exists a constant C_1, independent of p and u, such that

$$\sum_j \left| \int_G b_j''(x) D_j u, u^{2q-1} dx \right| \leqq C_1 \left\{ \int_G u^{2q} \, dx \right\}^{1/2} \left\{ \int_G u^{2q-2} \sum_j |D_j u|^2 \, dx \right\}^{1/2}.$$

Applying the usual estimate $ab \leqq \varepsilon a^2 + (4\varepsilon)^{-1} b^2$, and choosing ε to be less than $\frac{C_0(2q-1)}{2}$, we find that

$$\mathrm{Re}(A u, u^{2q-1}) \geqq c_0/2 \int_G u^{2q-2} \sum_j |D_j u|^2 \, dx - C_2 \|u\|^{2q}_{L^{2q}(G)} ,$$

with a constant C_2 independent of u or p.

On the other hand, for real ξ, we have

$$\mathrm{Re}(A_p u + \xi u, u^{2q-1}) \geqq \xi (u, u^{2q-1}) - C_2 \|u\|^{2q}_{L^{2q}(G)} \geqq (\xi - C_2) \|u\|^{2q}_{L^{2q}(G)} ,$$

while by Hölder's inequality,

$$\mathrm{Re}((A_p + \xi I) u, u^{2q-1}) \leqq \|(A_p + \xi I) u\|_{L^{2q}(G)} \cdot \|u\|^{2q-1}_{L^{2q}(G)} .$$

Thus,

$$\|(A_q + \xi I) u\|_{L^{2q}(G)} \|u\|^{2q-1}_{L^{2q}(G)} \geqq (\xi - C_2) \|u\|^{2q}_{L^{2q}(G)} ,$$

and, cancelling off the common positive factor on both sides of the inequality, we obtain the conclusion of Lemma 22 with $K = C_2$.

Lemma 24. *Let A be a uniformly elliptic differential operator of order two with real coefficients on an open set G_0 of E^n with its top order coefficients uniformly continuous on G_0 and all the coefficients of A and its formal adjoint A' essentially bounded on G_0. Let G be a bounded open subset of G_0 at positive distance from the complement of G_0 with G uniformly regular of class C^2 and locally regular of class C^4.*

Then there exists a constant K such that for real $\xi > K$ and all p with $1 < p < \infty$, $(A_p + \xi I)^{-1}$ is defined and bounded on all of $L^p(G)$ and satisfies an inequality of the form

$$\|(A_p + \xi I)^{-1}\| \leqq C_p (\xi - K)^{-1},$$

with a constant C_p uniformly bounded on every compact sub-interval of $(1, \infty)$.

Proof of Lemma 24. Under our assumptions, A and A' satisfy the regularity condition (IV) of Section 3. Hence by Theorem 16, $(A_p + \xi I)^* = (A'_s + \xi I)$, where s is the conjugate exponent of p. Since the adjoint of an invertible operator is invertible and since A' also satisfies the hypothesis of Lemma 24, it suffices to consider p in the range $2 \leqq p < \infty$. By the convexity theorem

of M. Riesz and Theorem 10 of Section 3, however, it suffices to consider only even integers p.

By Lemma 23, if p is an even integer > 0, we have for a real-valued function u in $D(A_p)$,

$$\|(A_p + \xi I)u\|_{L^p(G)} \geqq (\xi - K)\,\|u\|_{L^p(G)}\,.$$

If u is a complex-valued function in $D(A_p)$, $u = u_1 + i\,u_2$ with u_1 and u_2 real-valued, $(Au + \xi u) = (Au_1 + \xi u_1) + i(Au_2 + \xi u_2)$, where each of the terms in a paranthesis is a real-valued function. Thus,

$$\|(Au + \xi u)\|^p_{L^p(G)} \geqq \|Au_1 + \xi u_1\|^p_{L^p(G)} + \|Au_2 + \xi u_2\|^p_{L^p(G)} \geqq$$
$$\geqq (\xi - K)^p \{\|u_1\|^p_{L^p(G)} + \|u_2\|^p_{L^p(G)}\} \geqq C_p^{-p}(\xi - K)^p \|u\|^p_{L^p}\,.$$

Taking p-th roots, one has

$$\|(Au + \xi u)\|_{L^p(G)} \geqq C_p^{-1}(\xi - K)\,\|u\|_{L^p(G)}\,.$$

For a bounded domain, however, we have already remarked that the fact that $(A_p + \xi I)$ is one-to-one already implies that $(A_p + \xi I)^{-1}$ exists on all of $L^p(G)$. From our last inequality, we then obtain

$$\|(A_p + \xi I)^{-1}\| \leqq C_p(\xi - K)^{-1},$$

and the remainder of our conclusions follow.

Lemma 25. *Let A be a second-order elliptic operator satisfying the hypotheses of Lemma 23, p a positive even integer. Then there exist constants C_0 and K_1 such that for real $\xi > K_1$, and all u in $D(A_p)$,*

$$\|u\|_{2m,p} \leqq C_p\,\|(A_p + \xi I)u\|_{0,p}\,.$$

Proof of Lemma 25. By Theorem 2 in Section 2,

$$\|u\|_{2m,p} \leqq C'_p\{\|A_p u\|_{0,p} + \|u\|_{0,p}\}\,.$$

By the Minkowski inequality,

$$\|A_p u\|_{0,p} \leqq \|(A_p + \xi I)u\|_{0,p} + \xi\|u\|_{0,p}\,,$$

while by Lemma 23,

$$C'_p(1 + \xi)\,\|u\|_{0,p} \leqq C''_p(C'_p + C'_p\xi)\,(\xi - K)^{-1}\|(A_p + \xi I)u\|_{0,p}\,.$$

Choosing $K_1 = K + 1$, we see that for $\xi \geqq K_1$,

$$\|u\|_{2m,p} \leqq C'_p + C''_p(C'_p\xi + C'_p)\,(\xi - K_1 + 1)^{-1}\|(A_p + \xi I)u\|_{0,p}\,,$$

establishing the conclusion of Lemma 25 with the constant

$$C_p = \sup_{\xi \geqq K_1}\{C'_p + C''_p(C'_p + C'_p\xi)\,(\xi - K_1 + 1)^{-1}\}\,,$$

which is finite.

Theorem 31. *Let A be an essentially real, uniformly elliptic differential operator of second order which satisfies the regularity condition* (IV) *of Section 3 on an open set G of E^n which is uniformly regular of class C^2 and locally regular*

of class C^4. Then for each p with $1 < p < \infty$, there exist constants C_p and K_p such that for real $\xi \geqq K_p$, $(A_p + \xi I)^{-1}$ exists on all of $L^p(G)$ and satisfies the inequality

$$\|(A_p + \xi I)^{-1}\| \leqq C_p(\xi - K_p)^{-1}. \tag{7.2}$$

Proof of Theorem 31. We choose a family of functions $\{\eta_k\}$ as described in Lemma 2 of Section 1, comprising a uniformly locally finite partition of the identity on G with supports of diameter less than a small positive constant δ. Since $(A_p + \xi I)$ has $(A_s' + \xi I)$ as its adjoint, $(s = p(p-1)^{-1})$, it follows as in the proof of Lemma 24 that we need only consider p with $2 \leqq p < \infty$. By the Riesz convexity theorem and Theorem 10 of Section 3, we need only to consider even positive integer values for p. For the remainder of the present proof, we shall assume therefore that p is a positive even integer, $p = 2q$.

By the argument of Lemma 24, without any boundedness assumption on the domain G, it follows from Lemma 23 that there exist constants K and C such that for all u in $D(A_p)$,

$$\|(A_p + \xi I)u\|_{0,p} \geqq C_p(\xi - K)\|u\|_{0,p}.$$

In particular, it follows that $(A_p + \xi I)$ is one-to-one. If we show that its range is the whole of $L^p(G)$, it will follow that $(A_p + \xi I)^{-1}$ exists and satisfies the inequality (7.2) for all values of p.

In order to prove that the range of $(A_p + \xi I)$ is the whole of $L^p(G)$, we begin with several preliminary remarks. It follows from Lemmas 23 and 25 that there exists a constant C_p such that for real ξ,

$$\|u\|_{2m,p} + (\xi - K)\|u\|_{0,p} \leqq C_p\|(A_p + \xi I)u\|_{0,p} \tag{7.3}$$

for u in $D(A_p)$, $\xi \geqq K$. The constant C_p in (7.3) is obtained as the sum of the two such constants obtained in Lemmas 24 and 25, and we must consider the latter more closely. It is a consequence of the proof of Lemma 23 that the constant C_p obtained there is independent of the domain G and depends only upon the exponent p, (and would be equal to 1 for real-valued functions u). The constant in the inequality of Lemma 25 depends upon that of Lemma 23 and also upon that of Theorem 2 of Section 2, the latter depending only upon the regularity constants of the domain G as well as properties of the coefficients of A on G. We verify that we may choose a family of balls in E^n of radius $\delta > 0$, each containing support of one of the functions of the sequence $\{\eta_k\}$, such that if $G_k = G \cap B_k$, where B_k is the k-th ball considered, then the inequality (7.3) holds uniformly with a fixed constant C_p on all the open sets G_k.

Let $A_{p,k}$ denote the realization of A under null Dirichlet boundary conditions in $L^p(G_k)$. By Lemma 24, since all the hypotheses which are given for A are also valid for its restriction to G_k, it follows that there exists a constant K such that for real $\xi \geqq K$, $(A_{p,k} + \xi I)^{-1}$ is well-defined and bounded on the whole of $L^p(G_k)$. (The uniformity of the constant K over k follows from an inspection of the proof of Lemma 23.) We construct an operator S from $L^p(G)$ to $W^{2,p}(G)$ by the formula

$$S_\xi u = \sum_k \eta_k(x)(A_{p,k} + \xi I)^{-1}(\eta_k u). \tag{7.4}$$

By Lemma 2 and the fact that the support of each η_k is contained in the interior of $\overline{G} \cap B_k$, we see that for each $\xi \geqq K$, S_ξ is a transformation from $L^p(G)$ into $W^{2,p}(G)$.

Let us consider the result of applying $(A_p + \xi I)$ to $S_\xi u$. We obtain

$$\begin{aligned}(A_p + \xi I)\, S_\xi u &= \sum_k (A_p + \xi I)\, \{\eta_k(x)\, (A_{p,k} + \xi I)^{-1}\, (\eta_k u)\} \\ &= \sum_k [\eta_k(x)\, (A_p + \xi I)\, (A_{p,k} + \xi I)^{-1} (\eta_k u) + \\ &+ \sum_{|\alpha| < 2m} c_{\alpha,k}(x)\, D^\alpha \{(A_{p,k} + \xi I)^{-1} (\eta_k u)\}]\,,\end{aligned}$$

where by Lemma 2, the coefficient functions $c_{\alpha,k}(x)$, which are independent of ξ and have compact supports in B_k, satisfy the inequality $\sum_k |c_{\alpha,k}(x)| \leqq M_\alpha$. Since

$$(A_p + \xi I)\, (A_{p,k} + \xi I)^{-1} (\eta_k u) = \eta_k u$$

on G_k, it follows that

$$(A_p + \xi I)\, (S_\xi u) = \sum_k (\eta_k)^2 u + R_\xi(u) = u + R_\xi u\,,$$

where $R_\xi(u)$, being defined by

$$R_\xi(u) = \sum_k \sum_{|\alpha| < 2m} c_{\alpha,k}(x)\, D^\alpha \{(A_{p,k} + \xi I)^{-1} (\eta_k u)\}\,,$$

satisfies an inequality of the form

$$\|R_\xi(u)\|_{0,p} \leqq M \sup_k \|(A_{p,k} + \xi I)^{-1} (\eta_k u)\|_{2m-1,p}\,. \tag{7.5}$$

(The norms on the right of (7.5) are of course taken over the subsets G_k.)

Let $v_k = (A_{p,k} + \xi I)^{-1} (\eta_k u)$. Then $(A_{p,k} + \xi I) v_k = \eta_k u$. If ξ is chosen larger than K, as described above, uniformly in k, we have on G_k the inequality

$$\|v_k\|_{2m,p} + (\xi - K + 1)\, \|v_k\|_{0,p} \leqq \|\eta_k u\|_{0,p}\,.$$

On the other hand, by Lemma 5 of Section 1, we know that for any $\varepsilon > 0$, there exists a constant C_ε, independent of k by the uniform regularity of the domains G_k, such that

$$\|v_k\|_{2m-1,p} \leqq \varepsilon \|v_k\|_{2m,p} + C_\varepsilon \|v_k\|_{0,p}\,.$$

Choosing ε sufficiently small so that $M\varepsilon$ is less than $1/2$, and then choosing K_1 so large that $(\xi - K + 1) \geqq \varepsilon^{-1} C_\varepsilon$, we will have from the above inequalities

$$\begin{aligned}M \|v_k\|_{2m-1,p} &\leqq M\varepsilon \|v_k\|_{2m,p} + M C_\varepsilon \|v_k\|_{0,p} \leqq \\ &\leqq M\varepsilon \{\|v\|_{2m,p} + (\xi - K + 1)\, \|v\|_{0,p}\} \leqq M\varepsilon \|\eta_k u\|_{0,p} \leqq M\varepsilon \|u\|_{0,p}\,.\end{aligned}$$

Taking the supremum over k, we see that $\|R_\xi(u)\|_{0,p} \leqq 1/2\, \|u\|_{0,p}$. It follows that the transformation $(I + R_\xi)$ maps $L^p(G)$ onto itself for $\xi \geqq K_1$, where K_1 is chosen as above. We know moreover that $(A_p + \xi I) S_\xi$ is defined on all of $L^p(G)$ and equals $(I + R_\xi)$. For v in $L^p(G)$, let $u = (I + R_\xi)^{-1} v$. Then

$$(A_p + \xi I) S_\xi u = (I + R_\xi) u = v\,.$$

Since $S_\xi u$ lies in $D(A_p + \xi I)$, it follows that for $\xi \geqq K_1$, $(A_p + \xi I)$ has the whole of $L^p(G)$ as its range, and the conclusion of Theorem 31 follows.

We may modify the above proof by using a partition $\{\eta_k\}$ with small supports (i.e. choosing the diameter of the support small), and then applying the inequality of Lemma 24 to the constant coefficient operator $A_{0,k}$ obtained from A by evaluating its coefficients at a given point of each G_k. The same argument as we have given then yields the following result:

Theorem 32. *Let A be a second-order linear elliptic operator of second order which is essentially real and uniformly elliptic on an open subset G of E^n which is uniformly regular of class C^2. Suppose that the coefficients of A satisfy the regularity conditions* (I) *and* (II) *of Section* 2, *(i.e. top-order coefficients uniformly continuous and all other coefficients essentially bounded). Then for $2 \leqq p < \infty$, there exist constants C_p and K_p such that for $\xi \geqq K_p$, $(A_p + \xi I)^{-1}$ will exist and satisfy the inequality $\|(A_p + \xi I)\|^{-1} \leqq C_p(\xi - K_p)^{-1}$.*

We shall not give any further details in the proof of Theorem 32 both because of its similarity to the proof of Theorem 31 and that of Theorem 16, as well as the fact that we shall carry through a proof of similar type for a sharper result for equations of arbitrary even order in the next paper of this series.

We close this Section with a brief discussion of a partially strengthened form of the results of Section 5 on operators with potentials which may be given for second-order operators and which for second order operators on the whole of E^n is due to WIENHOLTZ [70].

Theorem 33. *Let A be a second-order elliptic differential operator with real coefficients on a domain G of E^n which is uniformly regular of class C^2 and locally regular of class C^4. Suppose that:*

(a) *A is formally self-adjoint.*

(b) *The top-order coefficients of A, $A_{jk}(x)$ are uniformly bounded on G.*

(c) *A is semi-bounded, i. e. $(Au, u) \geqq k_0(u, u)$ for a fixed k_0 and all u in $D(A_2) \cap R(G)$.*

(d) *On every bounded subset of G, A satisfies the regularity conditions* (IV) *of Section* 3.

Then: The restriction of A to $D(A_2) \cap R(G)$ is essentially self-adjoint in $L^2(G)$.

Proof. By adding a constant to A, we may assume that $k_0 = 1$. It suffices to show that the range of A (considered as an operator whose domain is the set of functions with compact support in $D(A_2)$) is a dense subset of $L^2(G)$. Let u be a function in the orthogonal complement of the range of A, i.e.

$$(u, Av) = 0\,,$$

for all v in $D(A_2)$ with compact support.

Let $n(x)$ be a real-valued function from $C_c^\infty(E^n)$ which is 1 for $|x| \leqq 1$, 0 for $|x| \geqq 2$, and for each $\varepsilon > 0$, let $n_\varepsilon(x) = n(\varepsilon^{-1}x)$. By the results of Section 3, u lies in $W^{2,2}_{\text{loc}}(G \cup \Gamma)$, where Γ is the boundary of G, (since it lies in the domain of $A_{2,1}$ for all bounded portions Γ_0 of Γ).

By the semi-boundedness condition (c) with $k_0 = 1$, for every $\varepsilon > 0$,

$$(n_\varepsilon u, n_\varepsilon u) \leqq (n_\varepsilon u, A(n_\varepsilon u)),$$

while by the calculations of WIENHOLTZ ([70], bottom of page 60),

$$n_\varepsilon u\, \overline{A(n_\varepsilon u)} = n_\varepsilon^2 u\, \overline{Au} + \sum_{j,k} |u|^2 a_{jk} (D_j(n_\varepsilon)\, \overline{D_k(n_\varepsilon)}) -$$

$$- \sum_{j,k} D_k(|u|^2 a_{jk} n_\varepsilon \overline{(D_j n_\varepsilon)}) + \sum_{j,k} a_{jk} n_\varepsilon \overline{(D_j n_\varepsilon)}\, (\bar{u}\, D_k u + u\, \overline{D_k u}) +$$

$$+ 2 \sum_{k=1}^{n} |u|^2 b_k n_\varepsilon \overline{D_k n_\varepsilon}.$$

Integrating over G, we see that $(n_\varepsilon u, A(n_\varepsilon u))$ is the integral of the left-side of the equation. On the right, the first term disappears when integrated as does the third. The last two terms are imaginary and therefore their sum must vanish when integrated. Finally, one obtains (since $|D_j n_\varepsilon| \leqq K\varepsilon$ on G),

$$\|n_\varepsilon u\|_{0,2}^2 \leqq K\varepsilon \|u\|^2.$$

Letting $\varepsilon \to 0$, $n_\varepsilon u \to u$ in $L^2(G)$, from which it follows that $u = 0$ and the proof is complete.

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